Finance Stochast. 8, 87–109 (2004)
DOI: 10.1007/s00780-003-0102-7

Finance and Stochastics

On the use of measure-valued strategies in bond markets

Marzia De Donno, Maurizio Pratelli

Dipartimento di Matematica, Università di Pisa, via F. Buonarroti 2, 56127 Pisa, Italy
(e-mail: {mdedonno;pratelli}@dm.unipi.it)

Abstract. We propose here a theory of cylindrical stochastic integration, recently developed by Mikulevicius and Rozovskii, as mathematical background to the theory of bond markets. In this theory, since there is a continuum of securities, it seems natural to define a portfolio as a measure on maturities. However, it turns out that this set of strategies is not complete, and the theory of cylindrical integration allows one to overcome this difficulty. Our approach generalizes the measure-valued strategies: this explains some known results, such as approximate completeness, but at the same time it also shows that either the optimal strategy is based on a finite number of bonds or it is not necessarily a measure-valued process.

Key words: Bond markets, term structure of interest rates, measure-valued portfolio, cylindrical stochastic integration, covariance spaces, market completeness

JEL Classification: G10, E43

Mathematics Subject Classification (1991): 60H05, 60G60, 90A09

1 Introduction

"In the continuous-time bond market model there is naturally a continuum of basic traded securities (zero-coupon bonds parameterized by their maturities) while in the standard model of stock market there is normally only a finite number of securities."

This statement, taken from [2], suggests to accept as admissible (though impossible to achieve in real markets) even portfolios containing an infinite number

The first author gratefully acknowledges financial support from the CNR Strategic Project "Modellizzazione matematica di fenomeni economici". We thank professors A. Bagchi, R. Douady and J. Zabczyk for helpful discussions. A special thanks goes to professors T. Björk, Y. Kabanov and W. Schachermayer for comments and suggestions which contributed to improve the final version of this paper.

Manuscript received: September 2001; final version received: November 2002

of traded securities. This gives rise to the problem of what exactly should be the meaning of the word "portfolio", at least in mathematical terms, or what processes should be considered the relative strategies. It seems natural to consider as a theoretical strategy, to be approximated by real strategies, a stochastic process with values in a set of measures on the time set (see, for instance, [14]). The question is whether there exists a satisfactory integration theory, which allows, from a mathematical point of view, to treat interest rate models like stock market models.

A first answer has been given by Björk et al. [2], who suggest two constructions of a stochastic integral where the integrator process takes values in the space of continuous functions and the integrand process (the portfolio strategy) is a measure-valued process.

In this paper, we address this problem, making use of a recently developed theory on cylindrical integration, due to Mikulevicius and Rozovskii [9, 10]. In fact, the construction of this integral gives a precise meaning to such a process, namely, a portfolio as an integral of measure-valued processes, and it also reveals that if we consider only measure-valued processes, the space of integrands is not complete. It is necessary to extend the notion of integrands to some processes which take values in a Hilbert space, called "covariance space" by Mikulevicius and Rozovskii.

This also allows to understand well a result due to Björk et al. [2] (see also [3]) which relates uniqueness of the martingale measure to "approximate completeness", namely, the possibility of approximating every (sufficiently integrable) claim with a sequence of portfolios based on measure-valued strategies. If the equivalent martingale measure is unique, a perfect hedging can be obtained by portfolios based on strategies which take values in the covariance space (this is just a straightforward application of Jacod-Yor Lemma, together with an extension of a representation theorem of square integrable martingales as stochastic integrals to the infinite-dimensional setting).

In some sense, we obtain a negative result: in fact we show that measure-valued strategies are sufficient to describe all possible portfolios only when the covariance spaces have finite dimension, and, in this case, all strategies are "true" strategies, namely, based on a finite number of bonds. Furthermore, the dimension of the covariance space characterizes the minimal number of bonds (which varies with (s,ω)) which are in the best portfolio. In particular, if all covariance spaces have dimension smaller than n, it would be completely useless (in the sense that it cannot improve the performance of an investor) to have more than n bonds in a portolio. It should be pointed out, however, that the maturities of these bonds are not fixed, but depend on (s,ω).

The paper is structured as follows. In Sect. 2, we recall some essential results from functional analysis: in particular, we illustrate, for the set of continuous functions on an interval, the main results on "reproducing kernels" of Schwartz [15], which is at the basis of the theory developed by Mikulevicius and Rozovskii. The Sect. 3 is devoted to briefly describe the main steps which lead to the construction of the cylindrical stochastic integral, and of a good class of integrands. Once again, we limit ourselves to the special case, which will be of use in financial applications, of integrals with respect to martingales in the space of continuous functions.

In both these sections, we omit (but give precise references) the proofs of theorems which can be found elsewhere and we develop only the proofs of new results. In Sect. 4, we apply the previous theory to bond market models, pointing out the consequences of the results we have obtained. Finally, in Sect. 5, we analyze a class of models introduced by Kennedy [7, 8], who assumes the forward rate curve to evolve as a Gaussian field: the covariance space can then be characterized uniquely by the covariance of the Gaussian field. We prove that Kennedy's model is approximately complete, in the sense that every contingent claim can be approximated by a sequence of measure-valued portfolios; however, each claim which depends on a finite number of bonds can be replicated by a portfolio which is based exactly on those bonds.

We wish to make it clear that this is only the first step in the use of cylindrical stochastic integration in bond market models: we restrict ourselves to the so called "martingale modelling" procedure (see, for instance, [1], p. 253), since we at the present can only deal with (locally square integrable) infinite-dimensional martingales. To proceed further, it would be, for instance, really important to clarify the notion of "cylindrical semimartingale".

2 Preliminaries on functional analysis

Let X be a compact metric space. Consider the space $\mathcal{C} = C(X)$ of continuous functions on X with the topology of uniform convergence; its topological dual is the space $\mathcal{M} = \mathcal{M}(X)$ of Radon measures on X, provided with the weak topology $\sigma(\mathcal{M}, \mathcal{C})$ with respect to which $\mathcal{M}$ is separable.

According to the terminology of [15], a *kernel* $\bar{Q}$ on $\mathcal{C}$ is a linear weakly continuous (hence continuous) function from $\mathcal{M}$ to $\mathcal{C}$. Denote by $\mathcal{L}^+(\mathcal{M}, \mathcal{C})$ the space of symmetric, non-negative definite kernels from $\mathcal{M}$ to $\mathcal{C}$, namely, the kernels $\bar{Q}$ such that for all $\mu, \nu \in \mathcal{M}$, $\langle \mu, \bar{Q}\nu \rangle = \langle \nu, \bar{Q}\mu \rangle$ and $\langle \mu, \bar{Q}\mu \rangle \geq 0$, where $\langle \ , \ \rangle$ denotes the canonical duality bilinear form.

Let $\bar{Q} \in \mathcal{L}^+(\mathcal{M}, \mathcal{C})$: the *reproducing kernel* generated by $\bar{Q}$ is the function $Q : X \times X \to \mathbb{R}$, defined by:

$$Q(x, y) = \langle \delta_x, \, \bar{Q}\delta_y \rangle.$$

Denote by $\mathcal{K}^+(X)$ set of symmetric, non-negative definite functions on $X \times X$, i.e. the functions $F : X \times X \to \mathbb{R}$ such that $F(x, y) = F(y, x)$ for all $x, y \in X$ and $\sum_{i \leq d} c_i F(x_i, x_j) c_j \geq 0$ for all $x_1, \ldots, x_d \in X, c_1, \ldots, c_d \in \mathbb{R}$, $d \in \mathbb{N}$. Then, it is easy to see that $Q \in \mathcal{K}^+(X)$. Furthermore, the function Q is separately continuous and bounded on $X \times X$.

Conversely, let $Q \in \mathcal{K}^+(X)$ be separately continuous, continuous on the diagonal and bounded; then, it is jointly continuous (see [15] for details). We can define a kernel $\bar{Q} \in \mathcal{L}^+(\mathcal{M}, \mathcal{C})$ by setting

$$\bar{Q}\mu(\cdot) = \int_X \mu(dx) Q(x, \, \cdot \,);$$

then, obviously, $\langle \nu, \bar{Q}\mu \rangle = \int_X \nu(dy) \int_X \mu(dx) Q(x, y)$.

For a given kernel $\bar{Q} \in \mathcal{L}^+(\mathcal{M}, \mathcal{C})$, a scalar product on $\bar{Q}(\mathcal{M})$ can be defined by:

$$(\bar{Q}\mu, \bar{Q}\nu) = \langle \mu, \bar{Q}\nu \rangle = \langle \nu, \bar{Q}\mu \rangle. \tag{1}$$

Proposition 2.1 *([15], Proposition 10) The set $\bar{Q}(\mathcal{M})$ admits a (unique) completion H_Q in $\mathcal{C}$, with respect to the norm induced by (1). This completion H_Q is a separable Hilbert space and can be continuously embedded in $\mathcal{C}$.*

A mapping is thus defined from $\mathcal{L}^+(\mathcal{M}, \mathcal{C})$ to the set of Hilbert subspaces of $\mathcal{C}$. It was proved by Schwartz ([15], Sects. 5–6) that it is in fact an isomorphism; in particular, each Hilbert subspace in $\mathcal{C}$, more precisely, each Hilbert space which can be continuously embedded in $\mathcal{C}$, is the image by a proper kernel $\bar{Q} \in \mathcal{L}^+(\mathcal{M}, \mathcal{C})$.

Remark Let Q be the reproducing kernel generated by $\bar{Q}$: it is a symmetric and non-negative definite function on $X \times X$, hence it is the covariance of a Gaussian process ([12], Proposition I.3.7). We then notice that H_Q is a well-known object: it is the reproducing kernel Hilbert space associated to the covariance Q ([12], pp. 37–38), namely, H_Q is the Hilbert space of functions on X, which is the closure of the subspace spanned by $\{Q(x, \cdot), x \in X\}$ (with respect to the topology induced by (1)) and it is such that $(h, Q(x, \cdot))_{H_Q} = h(x)$, for every $h \in H_Q$.

For the sake of simplicity, we will use the same notation for Q and $\bar{Q}$: it will be clear from the context whether Q refers to the element of $\mathcal{L}^+(\mathcal{M}, \mathcal{C})$ or of $\mathcal{K}^+(X)$.

Example 2.1 Let Q be of the form $Q(x, y) = \sum_{i \le n} a_i(x) a_i(y)$ with $a_i \in \mathcal{C}$. If the functions a_i are linearly independent, then $H_Q = span(a_1, \dots, a_n)$.

Let us consider in detail the case $n = 2$. Because of linear independence, we can find $\mu_1, \mu_2 \in \mathcal{M}$ such that $\int a_i d\mu_i = 1, \int a_i d\mu_j = 0$ for $i, j = 1, 2, i \neq j$. Then

$$a_i = Q\mu_i, \qquad (a_1, a_2)_{H_Q} = 0, \qquad |a_i|^2_{H_Q} = 1.$$

Furthermore, for any $\mu \in \mathcal{M}$, we have

$$Q\mu(\cdot) = a_1(\cdot) \int a_1 d\mu + a_2(\cdot) \int a_2 d\mu$$

which means that $Q\mu \in span(a_1, a_2)$, and $|Q\mu|^2_{H_Q} = \left(\int a_1 d\mu\right)^2 + \left(\int a_2 d\mu\right)^2$.

One can also prove the converse, which however will not be used in this paper: when the space H_Q has dimension n, then Q has the form $Q(x, y) = \sum_{i \le n} a_i(x) a_i(y)$ where a_i are continuous and linearly independent functions.

Example 2.2 Let $X = [0, 1]$ and $Q(x, y) = \min(x, y)$, the covariance of the Wiener process. In this case ([12], pp. 20–21, 37–38), the set H_Q is the subspace of $\mathcal{C}$ of functions h such that $h(0) = 0$, h is absolutely continuous and its derivative h' exists a.e. and belongs to $L^2(0, 1)$; furthermore, $|h|^2_{H_Q} = |h'|^2_{L^2(0,1)}$.

Indeed, given $\mu \in \mathcal{M}$,

$$Q\mu(y) = \int_0^1 \mu(dx) \min(x, y) = \int_0^y du\, \mu(u, 1] = h(y)$$

where $h'(y) = \mu(u, 1], h(0) = 0$;

$$|Q\mu|^2_{H_Q} = \langle Q\mu, \mu\rangle = \int_0^1 \mu(dy)\, Q\mu(y) = \int_0^1 du\, (\mu(u, 1])^2 = |h'|^2_{L^2(0,1)}$$

and moreover in general, for $\nu \in \mathcal{M}, g'(u) = \nu(u, 1], g(0) = 0$, $(Q\mu, \nu) = (h', g')_{L^2(0,1)}$.

The set H_Q is isomorphic to the subspace $\tilde{H}^1$ of the usual Sobolev space $H^1(0, 1) = W^{1,2}(0, 1)$ (see, for instance, [4]) of functions which vanish at 0. In particular it contains $H^1_0(0, 1)$.

It is not difficult to check that, if we consider $Q(x, y) = 1 + \min(x, y)$, we get

$$Q\mu(y) = \mu([0, 1]) + \int_0^y du\, \mu(]u, 1]).$$

In this case H_Q is exactly H^1.

The kernel $Q : \mathcal{M} \to \mathcal{C}$ is in fact a mapping from $\mathcal{M}$ to H_Q and $Q(\mathcal{M})$ is dense in H_Q. We want to investigate when $Q(\mathcal{M})$ is exactly H_Q. For instance, this happens in Example 2.1, but not in Example 2.2: in the latter case $Q(\mathcal{M})$ consists of all functions f such that $f(x) = \int_0^x g(t)dt$ where g has bounded variation, and hence it is strictly contained in H_Q.

It is easy to see that when H_Q has finite dimension, then $Q(\mathcal{M}) = H_Q$. We will show that, if Q is continuous (which, however, happens in most interesting examples), the converse is also true.

Theorem 2.2 *Assume that Q is continuous and $Q(\mathcal{M}) = H_Q$. Then* $\dim H_Q < \infty$.

Proof Consider $\mathcal{M}$ endowed with the total variation norm: the kernel Q is linear, continuous and onto H_Q. By theorem of Stone-Weierstrass, each continuous function $Q \in \mathcal{K}^+(X)$ can be uniformly approximated by functions of the form $\sum_{i\le n} a_i(x)a_i(y)$, with $a_i \in \mathcal{C}$. The kernel Q is a limit of a sequence of operators of finite rank, hence it is compact (see, for instance, [4]).

By the open mapping theorem ([4], Theorem II.5), for some $\varepsilon > 0$, we have $\varepsilon B_H \subset Q(U)$, where B_H is the closed unit ball in H_Q and U is the closed unit ball in $\mathcal{M}$ (with respect to total variation norm). Since Q is compact, $Q(U)$ is a compact set in H_Q. It follows that also B_H is compact in H_Q and, thus, H_Q has finite dimension ([4],Theorem VI.5). □

In Example 2.2, we have seen that H_Q can be equal to H^1. In general, this does not happen: if Q is sufficiently regular, then H_Q is a proper subspace in H^1 (and, as an obvious consequence, the dual of H^1 is contained in H'_Q). In particular, the following result holds ([15], Proposition 25):

Proposition 2.3 *If the kernel Q is $C^1(X \times X)$, then the embedding of H_Q in $C^1(X)$ is compact.*

Since $C^1(X)$ is continuously embedded in H^1, it follows that also the embedding of H_Q in H^1 is compact.

3 Integration with respect to a cylindrical martingale in the space of continuous functions

In this section, we shall recall briefly the theory developed by Mikulevicius and Rozovskii [9, 10], adapted to our particular setting. We refer to [6] for all basic definitions and notations.

Given a topological vector space E and its topological dual E', the pair (E, E') is called a *Schwartz pair* if E is locally convex and quasi-complete (i.e., all closed and bounded subsets of E are complete) and E' is weakly separable (see Definition 2.1 in [10]).

Suppose given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual assumptions and denote by $\mathcal{P}$ the predictable σ-field. We denote by $\mathcal{H}^2(\mathbb{P})$ (resp. $\mathcal{H}^2_{loc}(\mathbb{P})$) the set of square integrable martingales (resp. locally square integrable martingales).

A *cylindrical process* in E is a linear mapping on E' with values in a proper set of stochastic processes. In particular, a *locally square integrable cylindrical martingale in* E is a linear mapping $\mathbf{M} : E' \to \mathcal{H}^2_{loc}(\mathbb{P})$.

Let X be as in the previous section and denote by $\mathcal{B}(X)$ the Borel σ-algebra on X. Consider a family $\big((M^x_t)_{t\in[0,T]}\big)_{x\in X}$ of locally square integrable martingales (for all $x \in X$, $M^x \in \mathcal{H}^2_{loc}(\mathbb{P})$) and we make the following assumption:

Assumption 3.1 *There exist an increasing predictable process A_t and a function Q defined on $\Omega \times [0,T] \times X \times X$, measurable with respect to $\mathcal{P} \otimes \mathcal{B}(X) \otimes \mathcal{B}(X)$, such that:*

(i) the function $Q_{s,\omega}$ is in $\mathcal{K}^+(X)$ and is continuous, for all $(\omega, s) \in \Omega \times [0,T]$;
(ii) the function $\int_0^t Q_{s,\omega} dA_s(\omega)$, is in $\mathcal{K}^+(X)$ and is continuous, for all $(\omega, s) \in \Omega \times [0,T]$;
(iii) for fixed $x, y \in X$, for all $t \in [0,T]$,

$$< M^x, M^y >_t (\omega) = \int_0^t Q_{s,\omega}(x,y) dA_s(\omega) \qquad \textit{for } \mathbb{P}\textit{-a.e. } \omega.$$

We can define a linear mapping on the set of linear combinations of Dirac measures, which we denote by $\mathcal{D}$: for $\mu = \sum_{i\le n} c_i \delta_{x_i}$, we set

$$\mathbf{M}(\mu) = M^\mu_t = \sum_{i\le n} c_i M^{x_i}_t. \tag{2}$$

The process $\mathbf{M}(\mu)$ is a locally square integrable martingale; in fact the mapping $\mathbf{M}$ can be extended to a cylindrical martingale in $\mathcal{C}$. The pair $(\mathcal{C}, \mathcal{M})$ is a Schwartz pair; furthermore, Mikulevicius and Rozovskii proved ([9], Lemma 41) that for all $\nu \in \mathcal{M}$, there exists a sequence $\nu^n \in \mathcal{D}$ and a process $M^\nu \in \mathcal{H}^2_{loc}(\mathbb{P})$ such that

$$\sup_{0\le t\le T} |M^\nu_t - M^{\nu^n}_t| + \int_0^T \langle \nu - \nu^n, Q_s(\nu - \nu^n)\rangle \, dA_s \xrightarrow{\mathbb{P}} 0 \qquad (n \to \infty)$$

and $< M^\mu, M^\nu >_t = \int_0^t \langle Q_s \mu, \nu\rangle dA_s$, for all $\mu, \nu \in \mathcal{M}$.

The mapping $\mathbf{M}$ can be extended to a mapping on $\mathcal{M}$, still denoted by $\mathbf{M}$, with values in $\mathcal{H}^2_{loc}(\mathbb{P})$, and such that $\mathbf{M}(\nu) = M^\nu$ for $\nu \in \mathcal{M}$, where M^ν is defined as above. Thus, $\mathbf{M}$ is a locally square integrable cylindrical martingale in $\mathcal{C}$. According to the terminology of Mikulevicius and Rozovskii, the kernel Q is called *covariance operator function*, while the kernel $\int Q dA$ is called *predictable quadratic variation* for the cylindrical martingale $\mathbf{M}$.

Our aim is to give a sense of an integral of the form $\int F d\mathbf{M}$, hence to find a proper class of integrands. We shall follow the procedure of the construction due to Mikulevicius and Rozovskii. Let ν be a process with values in $\mathcal{M}$ of the form:

$$\nu_t(\omega) = \sum_{i=1}^d f_t^i(\omega)\mu_i \tag{3}$$

where $\mu_i \in \mathcal{M}$ and f^i are predictable bounded processes. Then, we set:

$$\mathcal{I}_t(\nu) = \int_0^t \nu_s d\mathbf{M}_s = \sum_{i=1}^d \int_0^t f_s^i dM_s^{\mu_i}; \tag{4}$$

Then $\mathcal{I}(\nu)$ is a locally square integrable martingale and its predictable quadratic variation is given by the formula:

$$< \mathcal{I}(\nu), \mathcal{I}(\nu) >_t = \int_0^t \langle \nu_s, Q_s \nu_s \rangle \, dA_s.$$

We denote by $L^2(\mathbf{M}, \mathcal{M})$ the set of processes ν such that ν takes values in $\mathcal{M}$, and is predictable, in the sense that for all $f \in \mathcal{C}$ the process $\langle \nu_s, f \rangle$ is predictable, and

$$\mathbb{E}\left[\int_0^T \langle \nu_s, Q_s \nu_s \rangle \, dA_s\right] < \infty. \tag{5}$$

The processes of the form (3) are dense in $L^2(\mathbf{M}, \mathcal{M})$, with respect to the norm (5). Then the map $\mathcal{I}$ defined by (4) can be uniquely extended to the set $L^2(\mathbf{M}, \mathcal{M})$, and by stopping also to the set $L^2_{loc}(\mathbf{M}, \mathcal{M})$, which is defined in the natural way. This extension, still denoted by $\nu \to \mathcal{I}(\nu)$, is linear, $\mathcal{I}(\nu) \in \mathcal{H}^2_{loc}(\mathbb{P})$ and

$$< \mathcal{I}(\nu), \mathcal{I}(\mu) >_t = \int_0^t \langle \nu_s, Q_s \mu_s \rangle dA_s$$

for all $\nu, \mu \in L^2_{loc}(\mathbf{M}, \mathcal{M})$. Furthermore, if $\nu^n, \nu \in L^2_{loc}(\mathbf{M}, \mathcal{M})$, and

$$\int_0^t \langle \nu_s^n - \nu_s, Q_s(\nu_s^n - \nu_s) \rangle \, dA_s \xrightarrow{\mathbb{P}} 0, \qquad (n \to \infty)$$

then,

$$\sup_{s \le t} |\mathcal{I}_s(\nu^n) - \mathcal{I}_s(\nu)| \xrightarrow{\mathbb{P}} 0 \qquad (n \to \infty).$$

This construction is however not satisfying since $L^2(\mathbf{M}, \mathcal{M})$ is not complete with respect to the norm defined by (5): a Cauchy sequence ν^n in $L^2(\mathbf{M}, \mathcal{M})$ does not necessarily converge to an element of that space. However, if ν^n is a Cauchy sequence in $L^2(\mathbf{M},\mathcal{M})$, then $\mathcal{I}(\nu^n)$ converges in $\mathcal{H}^2(\mathbb{P})$ to some square integrable martingale. In order to describe the limit process, we need to complete the set $L^2(\mathbf{M},\mathcal{M})$.

Let Q be the covariance operator function for the cylindrical martingale $\mathbf{M}$. For fixed (ω,t), $Q_{t,\omega}$ is in $\mathcal{L}^+(\mathcal{M},\mathcal{C})$: by Proposition 2.1, there exists a unique Hilbert space $H_{t,\omega}$, which can be continuously embedded in $\mathcal{C}$ and which is the completion of the set $Q_{t,\omega}(\mathcal{M})$ with respect to the norm induced by the scalar product:

$$(Q_{t,\omega}\mu, Q_{t,\omega}\nu) = \langle \mu, Q_{t,\omega}\nu\rangle.$$

In this way, we build a family $(H_{t,\omega})_{(t,\omega)\in[0,T]\times\Omega}$ of Hilbert subspaces in $\mathcal{C}$, which is called the *family of covariance spaces* for the cylindrical martingale $\mathbf{M}$.

Let h be a process on $\Omega\times[0,T]$, such that $h_t(\omega)\in H_{t,\omega}$ for all (t,ω); we say that h is *predictable* if the process $(\omega,t)\mapsto (h_t(\omega), Q_{t,\omega}\mu)_{H_{t,\omega}}$ is predictable, for any $\mu\in\mathcal{M}$.

We define the set

$$\widehat{L}^2(\mathbf{M},H) = \left\{ h\,\text{predictable} : \mathbb{E}\left[\int_0^T |h_s|^2_{H_s}\,dA_s\right] < \infty \right\}$$

which is clearly a Hilbert space; the set $\widehat{L}^2_{loc}(\mathbf{M},H)$ can be defined in the natural way. A mapping $\mathcal{Q}$ from $L^2_{loc}(\mathbf{M},\mathcal{M})$ to $\widehat{L}^2_{loc}(\mathbf{M},H)$ can be defined by associating to the process ν, the process h such that $h_t(\omega) = Q_{t,\omega}\nu_t(\omega)$ for all (t,ω). A new type of integral, called *normalized integral*, can be defined on $\mathcal{Q}(L^2_{loc}(\mathbf{M},\mathcal{M}))$, by setting

$$\int h * d\mathbf{M} = \int \nu d\mathbf{M}$$

for $h=\mathcal{Q}(\nu)$. The set $\mathcal{Q}(L^2_{loc}(\mathbf{M},\mathcal{M}))$ is dense in $\widehat{L}^2_{loc}(\mathbf{M},H)$ and the closure of $L^2_{loc}(\mathbf{M},\mathcal{M})$ is isometric to the space $\widehat{L}^2_{loc}(\mathbf{M},H)$, which is complete ([9], Proposition 10). It seems natural then to take this set as completion of the set of integrands: for all $h\in\widehat{L}^2_{loc}(\mathbf{M},H)$, there exists a sequence $\nu^n\in L^2_{loc}(\mathbf{M},\mathcal{M})$ such that

$$\int_0^t |Q_s\nu^n_s - h_s|^2_{H_s}\,dA_s \xrightarrow{\mathbb{P}} 0 \qquad (n\to\infty) \tag{6}$$

In particular, the sequence ν^n_s is a Cauchy sequence in $L^2_{loc}(\mathbf{M},\mathcal{M})$ and, as a consequence, the sequence $\mathcal{I}(\nu^n)$ is a Cauchy sequence in $\mathcal{H}^2_{loc}(\mathbb{P})$. We denote the limit of this sequence by $\int h * d\mathbf{M}$.

Remark The kernel $Q:\mathcal{M}\to\mathcal{C}$ can be continuously extended to the canonical isomorphism from H' to H, where H' is the topological dual of H. Moreover H' is the completion of the set $\mathcal{M}/ker\,Q$ with respect to the norm induced by the scalar product $(\mu,\nu)_{H'} = \langle\mu, Q\nu\rangle$. Then, denoting by $L^2_{loc}(\mathbf{M},H')$ the set of processes F which satisfy a weak predictability condition and such that the process $\left(\int_0^t |F_s|^2_{H'_s}dA_s\right)_{t\le T}$ are locally integrable, it turns out that the closure of $L^2(\mathbf{M},\mathcal{M})$ is exactly $L^2(\mathbf{M},H')$ and the mapping $\mathcal{Q}$ can be extended to an isomorphism from $L^2(\mathbf{M},H')$ to $\widehat{L}^2(\mathbf{M},H)$.

Example 3.1 Consider the Brownian sheet (Wiener noise in space and time) $\mathbf{W} = (W_t^x)_{(t,x)\in[0,T]\times[0,1]}$, namely, the Gaussian process with covariance:

$$Cov(W_t^x, W_s^y) = \min(x,y)\min(s,t).$$

We take $X = [0,1]$; then, $\mathbf{W}$ can be viewed as a cylindrical martingale on $C(X)$, on the time set $[0,T]$. It is easy to see that, for $x,y\in X$,

$$d < W^x, W^y >_t = \min(x,y)\,dt.$$

Hence, if we take $A_t = t$, we find that Q is as in Example 2.2. The set of (normalized) integrands consists in all predictable $\mathcal{C}$-valued functions h, such that $h_{s,\omega}(0) = 0$, $h'_{s,\omega}\in L^2(0,1)$ and

$$\int_0^T |h'_s|^2_{L^2(0,1)} dA_s < \infty.$$

When $h = \mathcal{Q}(\nu)$ for some $\nu \in L^2(\mathbf{W},\mathcal{M})$, namely, $h'_{s,\omega}(x) = \nu_{s,\omega}(x,1]$, we have

$$\int_0^t h_s * d\mathbf{W}_s = \int_0^t \nu_s d\mathbf{W}_s = \int_0^t \left(\int_0^1 \nu_s(dx)dW_s^x\right).$$

If $(N_t)_{t\le T}$ is a compensated Poisson process, with intensity 1, independent of W^x for all x, the cylindrical martingale $M_t^x = N_t + W_t^x$ admits as covariance operator function $Q_{s,\omega}(x,y) = 1+\min(x,y)$ and as covariance space $H^1(0,1)$, hence:

$$\widehat{L}^2_{loc}(\mathbf{M},H) = \left\{ h \text{ predictable} : \int_0^T |h_s|^2_{H^1} dA_s < \infty \ \mathbb{P}\text{-a.s.}\right\}.$$

We also notice that in this case, for all s, $H_s = H^1(0,1) = \mathbb{R}\oplus\tilde{H}^1$ (where $\tilde{H}^1$ has been defined in Example 2.2) and consequently $H'_s = \mathbb{R}\oplus(\tilde{H}^1)'$. Hence an integrand can be viewed, for the normalized integral, as a pair $k = (\alpha,h)$ with α predictable $\mathbb{R}$-valued process and h predictable and $\tilde{H}^1$-valued, or equivalently, for the standard integral, as $G = (\alpha, F)$, where F is predictable and $(\tilde{H}^1)'$-valued and:

$$\begin{aligned}\int_0^t k_s * d\mathbf{M}_s &= \int_0^t \alpha_s\,dN_s + \int_0^t h_s * d\mathbf{W}_s \\ &= \int_0^t \alpha_s\,dN_s + \int_0^t F_s\,d\mathbf{W}_s = \int_0^t G_s\,d\mathbf{M}_s.\end{aligned}$$

We denote by $\mathcal{S}(\mathbf{M})$ the stable subspace generated by $\mathbf{M}$ in the set of square integrable martingales, that is, the smallest closed set in $\mathcal{H}^2(\mathbb{P})$, stable for stochastic integration and containing all the M^x, possibly stopped (see, for instance, [11]). Then, the normalized integral is an isometry on $\mathcal{S}(\mathbf{M})$.

Though the following result is not explicitly stated by Mikulevicius and Rozovskii, it is just an extension to the infinite-dimensional case of the characterization of stable subspaces ([5], Theorem 4.35):

Proposition 3.1 *Each $N \in \mathcal{S}(\mathbf{M})$ can be represented in the form:*

$$N_t = N_0 + \int_0^t h_s * d\mathbf{M}_s \tag{7}$$

for some predictable h such that $\mathbb{E}\left[\int_0^T |h_s|^2_{H_s} dA_s\right] < \infty$.

We have seen that, in general, the set $Q(\mathcal{M})$ is a proper subset of H.

Thus, we cannot expect to obtain a representation of any element of $\mathcal{S}(\mathcal{M})$ as integral of a measure-valued process. This may however happen in some cases, depending on the structure of the covariance spaces.

Theorem 3.2 *Assume that for all (s, ω), dim $H(s, \omega) = n(s, \omega) < \infty$. Then any $N \in \mathcal{S}(\mathbf{M})$ can be written in the form:*

$$N_t = N_0 + \int_0^t \nu_s d\mathbf{M}_s \tag{8}$$

where $\nu_s(\omega)$ is a measure on X with finite support and the cardinality of the support is exactly $n(s, \omega)$.

The proof of this theorem is given in the appendix.

4 Applications to "bond markets"

We consider a model of bond markets based on a family of optional processes $(P(\cdot, T))_{T \leq T^*}$, which represent the bond prices for all maturities T. For basic definitions, assumptions and notations we refer mainly to [14] and [1]. Furthermore, we assume that $\mathcal{F}_0$ is the trivial σ-algebra.

We define the bank account $B_t = \exp\left(\int_0^t r_s ds\right)$, where r is the short rate process, and denote by $\overline{P}(t, T)$ the discounted bond process $P(t, T)/B_t$.

The process $P(t, T)$ is defined only for $t \leq T$. To work with processes which are defined for all t, we set $P(t, T) = \exp\left(\int_T^t r_s ds\right)$ for $t > T$, as already suggested in [2]. In economic terms, this means that we suppose that the owner of a zero coupon bond invests the money received at the bond maturity in the bank account.

In a bond market, unlike in a stock market, an investor can choose among a "continuum" of traded securities. For this reason, Björk et al. [2] suggest considering as possible strategies "measure-valued" processes; a rigorous definition can be found in [14] (Definition VII.5.2). According to this definition, a strategy $\pi = (\beta, \gamma)$ in a bond market consists of two processes:

(i) a predictable process $(\beta_t)_{t \leq T^*}$, which represents the quantity invested in the money market account;
(ii) a family of Radon measures $(\gamma_t)_{t \leq T^*}$ such that for all (t, ω), $(\gamma_t(dT))$ is a measure on $\mathcal{B}([0, T^*])$, with support concentrated on $[t, T^*]$ and for all $B \in \mathcal{B}([0, T^*])$, the process $(\gamma_t(B))_{t \leq T^*}$ is predictable; $\gamma_t(dT)$ can be interpreted as the "number" of bonds with maturity date in the interval $[T, T + dT]$.

The value of the portfolio generated by such a strategy is described by the process

$$V_t^\pi = \beta_t B_t + \int_t^{T^*} P(t,T)\gamma_t(dT).$$

The self-financing condition takes the form

$$dV_t^\pi = \beta_t dB_t + \left(\int_t^{T^*} dP(t,T)\gamma_t(dT) \right) \tag{9}$$

in a symbolic notation which needs to be given a certain meaning. In [14], an interpretation of this expression is given in the HJM framework. Björk, Kabanov, Runggaldier give a mathematical interpretation to (9) in [3] in the case of bonds driven by a marked point process (see also Example 4.2).

For the general definition, it seems to be necessary to have a stochastic integral for measure-valued processes with respect to a process with values in the set of continuous functions. In order to apply the results of the previous section, we need the following assumption:

Assumption 4.1 *There exists an equivalent martingale measure for the bond market, namely, a measure $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$, such that for every fixed $T \in [0, T^*]$, the discounted bond price process $(\overline{P}(t,T))_{0 \le t \le T^*}$ is a local martingale under $\widetilde{\mathbb{P}}$.*

The process $\overline{P}(\,\cdot\,, T)$ is not necessarily a true martingale (the interest rate may take negative values). However, we assume that $P(\,\cdot\,, T)$ is locally bounded, and, since B_t is continuous and strictly positive, for all T, $\overline{P}(\,\cdot\,, T)$ is a locally square integrable martingale.

Set $X = [0, T^*]$: we assume the existence of predictable processes A_t and Q_t, which satisfy Assumption 3.1, so that for all $T_1, T_2 \in X$

$$d < \overline{P}(\,\cdot\,, T_1), \overline{P}(\,\cdot\,, T_2) >_t (\omega) = Q_{\omega,t}(T_1, T_2) dA_t(\omega).$$

Notice that this requirement is fulfilled by all known models. We also observe that when $\min(T_1, T_2) \le t$, then $Q_{\omega,t}(T_1, T_2) = 0$.

We can associate to the family $(\overline{P}(\,\cdot\,, T))_{T \le T^*}$ a cylindrical martingale $\overline{\mathbf{P}}$ in $\mathcal{C}$. The theory developed in the previous section permits us to give a meaning to the integral of a measure-valued process, independently of the model. So, we can formulate the self-financing condition for a discounted portfolio, generated by a measure-valued strategy:

Definition 4.1 *A strategy $\pi = (\beta, \gamma)$ such that $\gamma \in L^2_{loc}(\overline{\mathbf{P}}, \mathcal{M})$, is self-financing if the discounted portfolio value $\overline{V}_t^\pi = V_t^\pi / B_t$ satisfies the condition:*

$$\overline{V}_t^\pi = V_0^\pi + \int_0^t \gamma_s \, d\overline{\mathbf{P}}_s.$$

The self-financing condition and the initial value of the portfolio uniquely determine the value of the money holding β. So we will specify a self-financing portfolio by the pair (V_0, γ).

We can give a financial interpretation to the results we have obtained in the previous sections. We assume that the filtration is generated by the processes $(P(\,\cdot\,, T))_{T \le T^*}$.

A *contingent claim* is an element of $L^2(\mathcal{F}, \widetilde{\mathbb{P}})$, that is, a square integrable random variable, with respect to the equivalent martingale measure, and measurable with respect to $\mathcal{F}$. We then introduce the following definition:

Definition 4.2 *A claim C is attainable if there exist $V_0 \in \mathbb{R}$ and $\gamma \in L^2(\overline{\mathbf{P}}, \mathcal{M})$, such that*

$$\frac{C}{B_{T^*}} = V_0 + \int_0^{T^*} \gamma_s \, d\overline{\mathbf{P}}_s.$$

From Theorem 2.2 and Proposition 3.1, it follows that the space of measure-valued strategies may be not satisfactory when discussing of hedging portfolios and completeness: we can find a sequence of measure-valued portfolios which converges in the set of locally square integrable martingales, but the limit process cannot be represented as a measure-valued portfolio. This means that measure-valued strategies are not sufficient to describe all possible portfolios in the market. For this reason we give a new notion of attainability (see also [2]):

Definition 4.3 *A claim C is asymptotically (or approximately) attainable if there exist $V_0 \in \mathbb{R}$ and $h \in \widehat{L}^2(\overline{\mathbf{P}}, H)$ such that*

$$\frac{C}{B_{T^*}} = V_0 + \int_0^{T^*} h_s * d\overline{\mathbf{P}}_s.$$

This means that there exists a sequence of self-financing measure-valued portfolios V^n such that $\overline{V}^n_{T^*}$ converges to C/B_{T^*} in $L^2(\mathcal{F}, \widetilde{\mathbb{P}})$.

In the given representation, h may not be the image of a measure, but Theorem 3.2 implies the following result:

Theorem 4.1 *Let C be an asymptotically attainable contingent claim. Assume that all the covariance spaces associated to Q have finite dimension: then, there exists a replicating strategy which is, in each point, a finite combination of Dirac measures.*

Note that the replicating portfolio may involve a continuum of bonds.

Remark One might imagine that the good strategy is the strategy based on integrands with values in $\big(H^1(0, T^*)\big)'$, but this is not possible. In fact, Q usually satisfies the assumptions of Proposition 2.3, hence for all (s, ω), the embedding of $H_{s,\omega}$ in $H^1(0, T^*)$ is compact and this implies that $\big(H^1(0, T^*)\big)'$ is a proper subspace of $H'_{s,\omega}$.

Example 4.1 Stochastic volatility models Assume that the zero coupon bond process evolves, under an equivalent martingale measure, according to a dynamics of the form:

$$dP(t,T) = P(t,T)\left(r(t)dt + \sum_{i=1}^{n} \sigma_i(t,\eta,T)dW_t^i\right)$$

where W^i are independent Wiener processes and η is an element of some measurable space $(E,\mathcal{E})$ (usually η is a stochastic process). Such models are called *stochastic volatility models*, driven by an n-dimensional Wiener process.

Computing the quadratic variation of the discounted process, we obtain:

$$d < \overline{P}(\cdot\,,T_1), \overline{P}(\cdot\,,T_2) >_t = \overline{P}(t,T_1)\overline{P}(t,T_2)\left(\sum_{i=1}^{n} \sigma_i(t,\eta,T_1)\sigma_i(t,\eta,T_2)\right) dt.$$

Then, omitting ω for simplicity, we can set

$$A_t = t \qquad\qquad Q_t(T_1,T_2) = \overline{P}(t,T_1)\overline{P}(t,T_2)\sum_{i=1}^{n} \sigma_i(t,\eta,T_1)\sigma_i(t,\eta,T_2).$$

We assume tha $\sigma_i(t,\eta,\cdot\,)$ is continuous for all fixed (t,η); then, Q has the same structure as in Example 2.1 and for all (t,ω), the associated covariance space has dimension less than or equal to n.

In most cases, these models are not complete. Theorem 4.1 tells us that for an investor it is not worthwhile to use at each point the whole rate curve, in the sense that the best performance can be obtained by a portfolio which, at each point (t,ω), has n bonds with different maturities and any more bond in the portfolio does not improve the replicating possibilities.

Example 4.2 Bond market in presence of marked point processes Consider a market where the bond prices are allowed to be driven by an n-dimensional Wiener process as well as a marked point process. This model has been carefully analyzed by Björk et al. ([3], p. 224), and we refer to them for assumptions and notations. They proved that, under an equivalent martingale measure, the discounted bond price dynamics is given by

$$d\overline{P}(t,T) = \overline{P}(t-,T)\left(\sum_{i=1}^{n} S_i(t,T)dW_t^i + \int_E \left(e^{D(t,x,T)} - 1\right)\tilde{\mu}(dt,dx)\right)$$

where W^i are independent Wiener processes, $\tilde{\mu}(dt,dx) = \mu(dt,dx) - \lambda_t(dx)dt$ is a compensated marked point process on $(E,\mathcal{E})$ (which is assumed to be a Lusin space) and λ is the intensity of μ under an equivalent martingale measure. The quadratic variation is:

$$d < \overline{P}(\cdot\,,T_1), \overline{P}(\cdot\,,T_2) >_t = \overline{P}(t-,T_1)\overline{P}(t-,T_2)\left[\sum_{i=1}^{n} S_i(t,T_1)S_i(t,T_2) + \int_E \left(e^{D(t,x,T_1)} - 1\right)\left(e^{D(t,x,T_2)} - 1\right)\lambda_t(dx)\right] dt.$$

Dropping t and ω, we can represent Q as follows:

$$Q(T_1, T_2) = \sum_{i=1}^{n} \alpha_i(T_1)\alpha_i(T_2) + \int_E \beta(x, T_1)\beta(x, T_2)\lambda(dx).$$

When λ is not concentrated at a finite number of points, the covariance space associated with Q has infinite dimension, and all possible strategies cannot be obtained using only measure-valued processes.

Our result allows us to explain the notion of "approximate completeness" and its relation with uniqueness of the martingale measure. We recall that a market is said to be complete if every contingent claim is attainable. In this setting, we need to extend this notion of completeness:

Definition 4.4 *The market is approximately (or asymptotically) complete if all claims are asymptotically attainable.*

Theorem 4.2 *If the martingale measure is unique, then the market is approximately complete.*

Proof By Proposition 3.1 and the well-known Jacod-Yor Lemma (see e.g., [5], Chap. XI), when the martingale measure is unique, each square integrable random variable C has a representation of the form

$$C = \mathbb{E}_{\widetilde{\mathbb{P}}}[C] + \int_0^{T^*} h_s * d\overline{\mathbf{P}}_s,$$

where $h \in \widehat{L}^2(\overline{\mathbf{P}}, H)$. In other words, any claim is asymptotically attainable. Hence, the market is approximately complete. □

This result has already been proved, with different techniques, by Björk et al. ([3], Proposition 4.7) and Björk et al. ([2], Proposition 6.11), for the the model considered in Example 4.2. As for the converse part, if the market is approximately complete, we can only deduce that the equivalent martingale measure is extremal in the set of all martingale measures. In [3], it is also shown that, in the case of Example 4.2, when λ is concentrated at a finite number of points, completeness and approximate completeness coincide. This is, in fact, the case where all the covariance spaces associated to Q have finite dimension. In this case, completeness is equivalent to uniqueness of the martingale measure.

A relation can be established between approximate completeness and completeness on a special set of claims.

Definition 4.5 *We say that a contingent claim C is a finite (market observable) contingent claim if there exists an integer d and maturities $T_1 < \ldots < T_d$ such that C is measurable with respect to the σ-algebra generated by the process $(P(\,\cdot\,, T_i))_{i \leq d}$.*

We shall see in the next section that, in the class of (infinite-dimensional) models introduced by Kennedy [7], such contingent claims can be hedged.

Theorem 4.3 *The market is approximately complete if and only if every finite contingent claim is asymptotically attainable.*

Proof Necessity is trivial. As for the converse part, it is not difficult to see (for instance by a Monotone Class Theorem) that the set of finite contingent claims is dense in $L^2(\mathcal{F}, \widetilde{\mathbb{P}})$, that is in the set of all contingent claims. □

5 Kennedy's model

We focus our attention on the forward-rate curve, assuming, for the sake of simplicity, that $T^* = 1$. Kennedy [7, 8] models the instantaneous forward rate $\{F(t,T) : 0 \le t \le T \le 1\}$ as a Gaussian field with independent increments in the t-direction. In particular, F has the form:

$$F(t,T) = \alpha(t,T) + Y(t,T), \qquad 0 \le t \le 1$$

where α is deterministic and continuous in t; $Y(t,T)$ is a centered continuous Gaussian random field with covariance structure specified by

$$Cov\,(Y(t_1,T_1), Y(t_2,T_2)) = c(t_1 \wedge t_2, T_1, T_2), \qquad 0 \le t_i \le T_i, \quad i = 1, 2, \tag{10}$$

where c is symmetric and non-negative definite in (T_1, T_2) and $c(0, T_1, T_2) = 0$. The dependence of c on $t_1 \wedge t_2$ ensures the independent increments property for Y in the t-direction: that is, the increment $Y(t_2,T) - Y(t_1,T)$ is independent of $\mathcal{F}_{t_1} = \sigma\{Y(u,v), u \le t_1, u \le v\}$, when $t_1 \le t_2 \le T$.

The zero-coupon bond price process at time t is given by:

$$P(t,T) = \exp\left(-\int_t^T F(t,s)ds\right);$$

the short rate is $r_s = F(s,s)$). Kennedy proved ([7], Theorem 1.1) that the discounted bond price $\overline{P}(t,T)$ is a martingale for all T if and only if

$$\alpha(t,T) = \alpha(0,T) + \int_0^T c(t \wedge s, s, T)ds.$$

We work directly under the martingale measure $\widetilde{\mathbb{P}}$. The discounted bond $\overline{P}(t,T)$ has the form $\zeta(t,T)\,\exp(D(t,T))$, where $\zeta(t,T)$ is a deterministic function, and

$$D(t,T) = -\int_t^T Y(t,s)ds - \int_0^t Y(s,s)ds. \tag{11}$$

The process $D(\,\cdot\,,T)$ is a martingale for all T and

$$dD(t,T) = \frac{1}{\overline{P}(t,T)} d\overline{P}(t,T). \tag{12}$$

Indeed, for $t_1 < t_2 \le T$,

$$D(t_2,T) - D(t_1,T) = -\int_{t_2}^{T} (Y(t_2,s) - Y(t_1,s))\,ds - \int_{t_1}^{t_2} (Y(s,s) - Y(t_1,s))\,ds.$$

Taking the conditional expectation with respect to $\mathcal{F}_{t_1}$ on both sides and exchanging the conditional expectation operator with the integral, we find that the right-hand side is zero; hence, $\mathbb{E}\left[D(t_2,T) - D(t_1,T)|\mathcal{F}_{t_1}\right] = 0$ and $D(\,\cdot\,,T)$ is a martingale. Moreover, by Ito's formula, we have

$$d\overline{P}(t,T) = e^{D(t,T)}\left(d\zeta(t,T) + \zeta(t,T)dD(t,T) + \frac{1}{2}\zeta(t,T)d < D(\,\cdot\,,T), D(\,\cdot\,,T) >_t\right).$$

Since $\overline{P}(t,T)$ is a martingale, the sum of all finite variation terms vanishes: it follows that $d\overline{P}(t,T) = \overline{P}(t,T)dD(t,T)$ or equivalently (12).

Lemma 5.1 *Let M be a continuous positive martingale, such that $M_t = a_{s,t}M_s$ for $s \le t$, where $a_{s,t}$ is independent of $\mathcal{F}_s$. Set $N_t = \int_0^t M_s^{-1} dM_s$ and assume N to be square integrable. Then N is a continuous martingale with independent increments and the process $< N, N >$ is deterministic.*

Proof The process N is a continuous martingale. The increment

$$N_t - N_s = \int_s^t M_u^{-1} dM_u$$

is the limit of processes of the form $\sum_{t_i \in \pi_n} M_{t_{i-1}}^{-1}(M_{t_i} - M_{t_{i-1}})$, where $(\pi_n)_n$ is a sequence of finite partitions of $[s,1]$, such that $mesh(\pi_n)$ tends to 0. By hypothesis, $\sum_{t_i \in \pi_n} M_{t_{i-1}}^{-1}(M_{t_i} - M_{t_{i-1}}) = \sum_{t_i \in \pi_n}(a_{t_{i-1},t_i} - 1)$; these random variables are independent of $\mathcal{F}_s$ and so is their limit. Finally, a square integrable continuous martingale N, with independent increments, has deterministic predictable quadratic variation: notice that this is just $< N, N >_t = \mathbb{E}\left[N_t^2\right]$. □

For all T, the process $\overline{P}(\,\cdot\,,T)$ satisfies the hypotheses of the lemma: it is continuous and $\overline{P}(t,T) = a_{s,t}\overline{P}(s,T)$, where

$$a_{s,t} = \exp\left(-\int_s^t (\alpha(u,u) - \alpha(s,u))\,du - \int_s^t (Y(u,u) - Y(s,u))\,du\right) \cdot \exp\left(-\int_t^T (\alpha(t,u) - \alpha(s,u))\,du - \int_t^T (Y(t,u) - Y(s,u))\,du\right).$$

Clearly, $a_{s,t}$ is independent of $\mathcal{F}_s$, since for all u, both $Y(u,u) - Y(s,u)$ and $Y(t,u) - Y(s,u)$ are independent of $\mathcal{F}_s$. By the lemma, $< D(\,\cdot\,,T), D(\,\cdot\,,T) >_t$ is deterministic and is explicitly computed as a function of the covariance of the

Gaussian process; the same can be done also with $< D(\cdot\,,T_1), D(\cdot\,,T_2) >_t$ (of course, $D(t,T_1)+D(t,T_2)$ is a square integrable continuous martingale with independent increments). Recalling that $D(t,T)$ is a martingale zero at 0, we have:

$$\begin{aligned}
& < D(\cdot\,,T_1), D(\cdot\,,T_2) >_t \\
&= \mathbb{E}\left[D(t,T_1)D(t,T_2)\right] = Cov(D(t,T_1), D(t,T_2)) \\
&= \int_0^t du \int_0^t ds\, c(s\wedge u, s, u) + \int_0^t du \int_t^{T_1} ds\, c(t\wedge u, u, s) \\
&\quad + \int_0^t du \int_t^{T_2} ds\, c(t\wedge u, u, s) + \int_t^{T_1} du \int_t^{T_2} ds\, c(t, u, s) \\
&= 2\int_0^t du \int_0^u ds\, c(s, s, u) + \int_0^t du \int_t^{T_1} ds\, c(u, u, s) \\
&\quad + \int_0^t du \int_t^{T_2} ds\, c(u, u, s) + \int_t^{T_1} du \int_t^{T_2} ds\, c(t, u, s).
\end{aligned}$$

From (12), it follows that

$$d < D(\cdot\,,T_1), D(\cdot\,,T_2) >_t = \frac{d < \overline{P}(\cdot\,,T_1), \overline{P}(\cdot\,,T_2) >_t}{\overline{P}(t,T_1)\overline{P}(t,T_2)}.$$

Suppose that there exist processes Q and A, such that

$$d < \overline{P}(\cdot\,,T_1), \overline{P}(\cdot\,,T_2) >_t = \overline{P}(t,T_1)\overline{P}(t,T_2)\, Q_t(T_1,T_2) dA_t;$$

then, we shall have

$$< D(\cdot\,,T_1), D(\cdot\,,T_2) >_t = \int_0^t Q_s(T_1,T_2) dA_s. \tag{13}$$

We now impose an additional hypothesis on the covariance function c: we take c to be of the form

$$c(t,T_1,T_2) = f(t) g(T_1,T_2),$$

where f is non-decreasing and $g(T_1,T_2)$ is symmetric and non-negative definite in (T_1,T_2). This is the case, for instance, when the random field of forward rates is Markov, in the sense of [8], Definition 3.3, and satisfies the independent increments property ([8], Theorem 3.1). Further, we suppose that f is $C^1(0,1)$ with $f'(t)>0$ for all t. Then both sides of (13) are differentiable in t and we easily find, by simple computations, that, for all $T_1, T_2 \geq t$:

$$\begin{aligned}
&\frac{d}{dt}\int_0^t Q_s(T_1,T_2)dA_s \\
&= 2\int_0^t ds\, f(s)g(s,t) + f(t)\int_t^{T_1} ds\, g(t,s) - \int_0^t du\, f(u)g(u,t) \\
&\quad + f(t)\int_t^{T_2} ds\, g(t,s) - \int_0^t du\, f(u)g(u,t) - f(t)\int_t^{T_2} du\, g(u,t) \\
&\quad - f(t)\int_t^{T_1} ds\, g(t,s) + f'(t)\int_t^{T_1} du \int_t^{T_2} ds\, g(u,s) \\
&= f'(t)\int_t^{T_1} du \int_t^{T_2} ds\, g(u,s).
\end{aligned}$$

Setting $dA_t = f'(t)dt$, we obtain

$$Q_t(T_1,T_2) = \int_t^{T_1} du \int_t^{T_2} ds\, g(u,s). \tag{14}$$

We have thus found A and Q satisfying Assumption 3.1. Let μ belong to $\mathcal{M}$: there exists an absolutely continuous function h on $[0,1]$, such that $h(0) = 0$ and $h'(s) = \mu(s,1]$ (see, for instance, [12]). If we define $k(s) = \mu(s,1]$, we see that k is in $L^2(0,1)$ and

$$\begin{aligned}
Q_t\mu(y) &= \int_t^1 \mu(dx)Q_t(x,y) = \int_t^1 \mu(dx)\int_t^x ds \int_t^y du\, g(s,u) \\
&= \int_t^1 ds\, \mu(s,1]\int_t^y du\, g(s,u) = \int_t^1 ds\, k(s)\int_t^y du\, g(s,u)
\end{aligned}$$

and now

$$\langle Q_t\mu, \mu\rangle = \int_t^1 ds \int_t^1 du\, k(s)k(u)g(s,u).$$

Example 5.2 Suppose that $g(s,u) = b(s)b(u)$. Then, we have

$$Q_t(T_1,T_2) = \left(\int_t^{T_1} ds\, b(s)\right)\left(\int_t^{T_2} du\, b(u)\right) = \beta_t(T_1)\beta_t(T_2).$$

Hence, the associated covariance space has dimension 1 (see Example 2.1). This is the case considered by Kennedy in [8], Theorem 3.3, when $\lambda = 2\mu$.

Example 5.3 Suppose that $g(s,u) = s \wedge u$. This situation corresponds to the case where Y is the Brownian sheet and $f(t) = t$. Let μ belong to $\mathcal{M}$ and h and k defined as above. Then we can explicitly compute $Q_t\mu$ and $\langle Q_t\mu, \mu\rangle$. For simplicity we consider $t = 0$. After some calculations, we obtain:

$$\begin{aligned} Q_0\mu(y) &= \int_0^1 ds\, k(s) \int_0^y du\,(s \wedge u) \\ &= -\int_0^y ds\, h(s)s + h(1)\frac{y^2}{2} \\ &= \int_0^y ds\tilde{h}(s)(y-s) \end{aligned}$$

$$\begin{aligned} \langle Q_0\mu, \mu\rangle &= \int_0^1 ds\, k(s) \int_0^1 du\, k(u)\,(s \wedge u) \\ &= \int_0^1 (h(1) - h(s))^2 ds \\ &= |\tilde{h}|^2_{L^2(0,1)} \end{aligned}$$

where $\tilde{h}(s) = h(1) - h(s)$. The covariance space associated to Q is isomorphic to the closure in $L^2(0,1)$ of the set of continuous functions $\tilde{h}$, such that $\tilde{h}(1) = 0$ and $\tilde{h}'$ is in $L^2(0,1)$. Since this set contains the space of functions on $[0,1]$ which are infinitely differentiable and with compact support, (which is dense in $L^2(0,1)$), the set H_Q is isomorphic to $L^2(0,1)$, hence it has infinite dimension.

From the previous examples, we see that this class of models includes both the cases where the covariance space has finite and infinite dimension; in the latter case, in order to hedge some claims, one may need "theoretical" portfolios, which are impossible to achieve in a real financial market. The class of finite contingent claims has an interesting property:

Theorem 5.1 *Let C be a finite (market observable) contingent claim, depending on the bonds $P(\cdot, T_1), \dots, P(\cdot, T_d)$, for some $d \in \mathbb{N}$, for some maturities $T_1 < \dots < T_d$. Then, it can be replicated with a portfolio based on $B, P(\cdot, T_1), \dots, P(\cdot, T_d)$.*

Proof By definition, $\overline{C} = C/B_1$ is measurable with respect to the σ-algebra generated by the $\mathbb{R}^d$-valued martingale $\tilde{D} = (D(\cdot, T_i))_{i \le d}$. By Lemma 5.1, $\tilde{D}$ is a continuous martingale with independent increments. Hence, by Theorem III.4.34 in [6], it has the predictable representation property; namely, all square integrable random variables $\overline{C}$, which are measurable with respect to the σ-algebra generated by $\tilde{D}$, can be written in the form:

$$\overline{C} = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\overline{C}\right] + \int_0^1 \varphi_t\, d\tilde{D}_t,$$

for some predictable process φ with values in $\mathbb{R}^d$ and integrable with respect to the martingale in $\tilde{D}$. The equality

$$\int_0^1 \varphi_t d\tilde{D}_t = \int_0^1 \left(\frac{\varphi_t^1}{\overline{P}(t,T_1)}, \dots, \frac{\varphi_t^d}{\overline{P}(t,T_d)}\right) d\left(\overline{P}(t,T_1), \dots, \overline{P}(t,T_d)\right)$$

shows that C is an attainable claim and the replicating portfolio is a self-financing portfolio based only on $B, P(\cdot, T_1), \ldots, P(\cdot, T_d)$. □

We would like to remark that this theorem holds in general, without any assumptions on c. What we have proved is that, in this class of models, a claim depending on a finite number of bonds can be hedged with a portfolio based exactly on those bonds and the bank account, and this is independent of the number of random sources which are involved, even when the market contains a continuum of random sources as, for instance, in the case of the Brownian sheet. Furthermore this result, together with Theorem 4.3, allows us to state the following:

Theorem 5.2 *Kennedy's bond market is approximately complete.*

6 Conclusions

The main purpose of this paper is the development of a mathematical technique, which permits one to use stochastic integration with respect to a family of "zero-coupon bonds", which can be seen as a stochastic process with values in the set of continuous functions. To this end, we adapt to this particular case the cylindrical stochastic integration theory introduced by Mikulevicius and Rozovskii in a more general setting [9, 10]. The class of integrands is interpreted, from a financial point of view, as the set of all possible strategies in the bond market. Björk et al. [2] have suggested considering "measure-valued portfolios"; we prove that, in order to obtain completeness, or even replication of some contingent claims, it is necessary to consider a larger set of strategies. As an example, we give a detailed analysis of a model introduced by Kennedy [7, 8]: in this case, we can explicitly characterize the set of strategies, which usually properly includes measure-valued processes. However, in this particular example, options depending on a finite number of bonds $P(t, T_1), \ldots, P(t, T_n)$ (as, for instance, swaps, caps and floors), can be replicated by using portfolios which are based on these n bonds and the money market account. We also prove that this model is complete.

Appendix: Proof of Theorem 3.2

We first observe that a compact metric space X is separable. For the sake of simplicity, we prove the theorem, assuming that X is an interval (this is, in fact, the only case we consider for applications): in particular, we take $X = [0, 1]$. The proof in the general case is easily obtained replacing the set $\mathbb{Q} \cap [0, 1]$ with a countable dense subset in X.

In the next two lemmas we will use the notations of Sect. 2.

Lemma A.1 *Let* $x_1, \ldots x_n \in [0, 1]$. *The set* $\{Q(x_i, \cdot)\}_{i \leq n}$ *is a basis for* H_Q *(and* $\dim H_Q = n$*) if and only if the following conditions hold:*

(i) det $[(Q(x_i, x_j))_{1 \leq i,j \leq n}] \neq 0$
(ii) det $[(Q(x_i, x_j))_{1 \leq i,j \leq n+1}] = 0$ *for all* $x_{n+1} \in [0, 1]$.

Proof Recalling that $(Q(x,\,\cdot\,), Q(y,\,\cdot\,))_{H_Q} = Q(x,y)$, it is easy to check that conditions (i) and (ii) hold if and only if $Q(x_1,\,\cdot\,),\ldots,Q(x_n,\,\cdot\,)$ are linearly independent and all $h \in H_Q$ are of the form $h = \sum_{i\le n} \lambda_i Q(x_i,\,\cdot\,)$. □

Remark When Q is continuous, condition (ii) can be replaced by

(ii') *det* $[(Q(x_i,x_j))_{1\le i,j\le n+1}] = 0$ *for all* $x_{n+1} \in \mathbb{Q}\cap[0,1]$.

Lemma A.2 *Assume that* Q *is continuous and* dim $H_Q = n$. *Then there exist* $x_1,\ldots x_n \in [0,1]\cap\mathbb{Q}$ *such that* $\{Q(x_i,\,\cdot\,)\}_{i\le n}$ *is a basis for* H_Q.

The proof of this lemma is easily obtained by using the continuity of Q and the Hilbert-Schmidt orthogonalization procedure. In the two lemmas which follow, we use the notations of Sect. 3 and suppose that Assumption 3.1 is fulfilled.

Lemma A.3 *Assume that dim* $H_{s,\omega} = n$, *and that there exist* $x_1,\ldots,x_n \in [0,1]$ *such that the set* $\{Q_{s,\omega}(x_i,\,\cdot\,)\}_{i\le n}$ *is a basis for* $H_{s,\omega}$, *for all* (s,ω). *Then,* $\mathcal{S}(M^{x_1}$, ..., $M^{x_n}) = \mathcal{S}(\mathbf{M})$.

Proof Set $\mathbf{M}^x = (M^{x_1},\ldots,M^{x_n})$. It will be sufficient to prove that $M^y \in \mathcal{S}(\mathbf{M}^x)$, for all $y \in [0,1]$: that is, there exists a measure-valued process ν of the form $\nu(s,\omega) = \sum_{i\le n} \lambda_i(s,\omega)\delta_{x_i}$, where $\lambda = (\lambda_i)_{i\le n}$ is a predictable process, such that the local martingale M^y has a representation of the type

$$M^y = \int \nu\, d\mathbf{M},$$

or, equivalently,

$$M^y = \int \lambda d\mathbf{M}^x.$$

We recall that λ is integrable with respect to the locally square integrable martingale $\mathbf{M}^x$ (and that the stochastic integral is still a locally square integrable martingale) if and only if the process

$$\sum_{i,j\le n} \int \lambda_i\lambda_j\, d < M^{x_i}, M^{x_j} >= \sum_{i,j\le n} \int \lambda_i\lambda_j\, Q(x_i,x_j)\, dA$$

is locally integrable (see, for instance, [6], Sect. III.4a).

Since $Q_{s,\omega}(y,\,\cdot\,) \in H_{s,\omega}$, by Lemma A.2, it can be written in the form

$$Q_{s,\omega}(y,\,\cdot\,) = \sum_{i=1}^{n} \lambda_i(s,\omega) Q_{s,\omega}(x_i,\,\cdot\,)$$

for some λ_i. In particular, for all j,

$$Q_{s,\omega}(y,x_j) = \sum_{i=1}^{n} \lambda_i(s,\omega) Q_{s,\omega}(x_i,x_j). \tag{15}$$

In virtue of Lemma A.1, λ is a solution of a system of linear equations with predictable coefficients, hence, it is predictable.

Set $\nu = \sum_{i \le n} \lambda_i \delta_{x_i}$ and $\tilde{\nu} = \delta_y - \nu$. Then, for all (s, ω)

$$|\tilde{\nu}(s,\omega)|^2_{H'_{s,\omega}} = Q_{s,\omega}(y,y) + \sum_{i,j=1}^{n} \lambda_i(s,\omega)\lambda_j(s,\omega) Q_{s,\omega}(x_i,x_j)$$
$$-2\sum_{i=1}^{n} \lambda_i(s,\omega) Q_{s,\omega}(x_i,x_j)$$

which is zero by (15). This implies that $\tilde{\nu}$ is $\mathbf{M}$-integrable and $\int \tilde{\nu} d\mathbf{M} = 0$. Since

$$\int \tilde{\nu}\, d\mathbf{M} = M^y - \int \nu\, d\mathbf{M},$$

it follows that ν is integrable with respect to $\mathbf{M}$ and $M^y = \int \nu\, d\mathbf{M} = \int \lambda\, d\mathbf{M}^x$ belongs to $\mathcal{S}(\mathbf{M}^x)$. □

Remark For a given predictable set B, we denote by $\mathbf{1}_B \cdot \mathbf{M}$ the cylindrical martingale associated to the family $(\mathbf{1}_B \cdot M^x)_{x \in [0,1]}$ (where, according to a standard notation, $\mathbf{1}_B \cdot M^x = \int \mathbf{1}_B\, dM^x$).

The previous lemma can be applied, with the proper restrictions to $\mathcal{S}(\mathbf{1}_B \cdot \mathbf{M})$.

Lemma A.4 *Let $x_1, \ldots, x_n \in [0,1]$. Denote by $B(x_1, \ldots, x_n)$ the set of all (s, ω) such that $\{Q_{s,\omega}(x_i, \cdot)\}_{i \le n}$ is a basis for $H_{s,\omega}$. Then, $B(x_1, \ldots, x_n)$ is a predictable set.*

Proof The set $B(x_1, \ldots, x_n)$ is the set of all (s, ω) such that

$$det\left[(Q_{s,\omega}(x_i, x_j))_{1 \le i,j \le n}\right] \ne 0$$

and

$$det\left[(Q_{s,\omega}(x_i, x_j))_{1 \le i,j \le n+1}\right] = 0$$

for all $x_{n+1} \in \mathbb{Q} \cap [0,1]$. Hence, it is predictable. □

Proof of Theorem 3.2 By Lemma A.2, $\Omega \times [0,T]$ is the (countable) union of all sets $B(x_1, \ldots x_n)$ as defined in Lemma A.4, for $n \in \mathbb{N}, x_1, \ldots, x_n \in [0,1] \cap \mathbb{Q}$. From this family of sets, we can construct a countable family of predictable and disjoint sets $(B_n)_{n \in \mathbb{N}}$, such that $\Omega \times [0,T] = \bigcup_n B_n$, and $H_{s,\omega}$ is generated by some $(Q_{s,\omega}(x_i^n, \cdot))_{1 \le i \le k_n}$, $x_i^n \in \mathbb{Q}$, for all $(\omega, s) \in B_n$.

Let N be an element of $\mathcal{S}(\mathbf{M})$. Then

$$N_t = \int_0^t h_s * d\mathbf{M}_s = \sum_{n \in \mathbb{N}} \int_{B_n \cap [0,t]} h_s * d\mathbf{M}_s,$$

for some predictable h such that $\mathbb{E}\left[\int_0^T |h_s|^2_{H_s} dA_s\right] < \infty$.

By the remark on Lemma A.3,

$$\int_{B_n \cap [0,t]} h_s * d\mathbf{M}_s = \int_{B_n \cap [0,t]} \lambda_s^n d\mathbf{M}_s^{x^n}$$

where $\mathbf{M}^{x^n} = (M^{x_1^n}, \ldots, M^{x_{k_n}^n})$ and $\lambda^n = (\lambda^1, \ldots, \lambda^{k_n})$ is a predictable process, integrable with respect to $\mathbf{1}_{B_n} \cdot \mathbf{M}^{x^n}$.

Put $\nu_s^n(\omega) = \sum_{i=1}^{k_n} \lambda_i^n(s,\omega)\delta_{x_i^n}$ on B_n, zero otherwise: clearly, ν^n is integrable with respect to $\mathbf{1}_{B_n} \cdot \mathbf{M}$, since

$$\mathbb{E}\left[\int_{B_n\cap[0,T]} |\nu_s^n|_{H_s'}^2 dA_s\right] = \mathbb{E}\left[\int_{B_n\cap[0,T]} |h_s|_{H_s}^2 dA_s\right] \leq \mathbb{E}\left[\int_0^T |h_s|_{H_s}^2 dA_s\right] < \infty.$$

Thus, the stochastic process ν which coincides with ν^n on B_n is well-defined and satisfies the requirements of the theorem, since

$$\int \nu\, d\mathbf{M} = \sum_n \int_{B_n} \nu\, d\mathbf{M} = \sum_n \int_{B_n} \nu^n d\mathbf{M} = \sum_n \int_{B_n} h * d\mathbf{M} = \int h * d\mathbf{M}.$$

□

References

1. Björk, T.: Arbitrage theory in continuous time. Oxford: Oxford University Press 1998
2. Björk, T., Di Masi, G., Kabanov, Y., Runggaldier, W.: Towards a general theory of bond markets. Finance Stochast. **1**, 141–174 (1997)
3. Björk, T., Kabanov, Y., Runggaldier, W.: Bond market structure in the presence of marked point processes. Math. Finance **7** (2), 211–239 (1997)
4. Brezis, H.: Analyse fonctionelle – Théorie et applications. Paris: Masson Editeur 1983
5. Jacod, J.: Calcul stochastique et problèmes de martingales (Lecture Notes in Mathematics **714**). Berlin Heidelberg New York: Springer 1979
6. Jacod, J., Shiryaev, A. N.: Limit theorems for stochastic processes. Berlin Heidelberg New York: Springer 1987
7. Kennedy, D. P.: The term structure of interest rates as a Gaussian random field. Math. Finance **4** (3), 247–258 (1994)
8. Kennedy, D. P.: Characterizing Gaussian models of the term structure of interest rates. Math. Finance **7** (2), 107–118 (1997)
9. Mikulevicius, R., Rozovskii, B. L.: Normalized stochastic integrals in topological vector spaces. In: Séminaire de probabilités XXXII (Lecture notes in Mathematics). Berlin Heidelberg New York: Springer 1998, 137–165
10. Mikulevicius, R., Rozovskii, B. L.: Martingales problems for stochastic PDE's. In: Carmona, R., Rozovski, B. (eds.) Stochastic partial differential equations: Six perspectives. (Math. Surveys and Monographs, **64**) Amer. Math. Soc. 1999, 243–326
11. Protter, P.: Stochastic integration and differential equations. Berlin Heidelberg New York: Springer 1990
12. Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Berlin Heidelberg New York: Springer 1991
13. Rudin, W.: Functional Analysis. New York: McGraw Hill 1973
14. Shiryaev, A. N.: Essentials of stochastic finance: Facts, models, theory. (Advanced Series on Statistical Science & Applied Probability, vol. 3). World Scientific 1999
15. Schwartz, L.: Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés (noyaux reproduisants). J. d'Analyse Math. **13**, 115–256 (1964)