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Super-replication and utility maximization in large financial markets

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Abstract

We study the problems of super-replication and utility maximization from terminal wealth in a semimartingale model with countably many assets. After introducing a suitable definition of admissible strategy, we characterize superreplicable contingent claims in terms of martingale measures. Utility maximization problems are then studied with the convex duality method, and we extend finite-dimensional results to this setting. The existence of an optimizer is proved in a suitable class of *generalized* strategies: this class has also the property that maximal expected utility is the limit of maximal expected utilities in finite-dimensional submarkets. Finally, we illustrate our results with some examples in infinite dimensional factor models.

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1. Introduction

A financial market is usually modeled as a d-dimensional semimartingale on a filtered probability space which describes the evolution of the discounted prices of d financial assets: we will refer to such a model as a *small market*, using a terminology introduced by Klein and Schachermayer [13]. The notion of *large financial market* was introduced by Kabanov and Kramkov [10] in order to describe a financial market containing a very large number of traded assets: they consider a *sequence* of $\mathbb{R}^{d(n)}$-valued semimartingales based on possibly different probability spaces. Several papers [10–13] investigate different notions of *asymptotic arbitrage* and also an extension of the *fundamental theorem of asset pricing* to the framework of large markets.

With this approach, a *large financial market* can be seen as a market where it is possible to choose a finite number of securities to trade, but a priori this number is not bounded. If we assume that all the probability spaces coincide, an alternative approach is to model a large financial market as a market where there is an infinite (countable) number of different assets available for trading, though in fact a *real* portfolio may only include a finite number of them: we will refer to such a *real* portfolio as an *elementary portfolio*.

In this spirit, Björk and Näslund [1] study a continuous time extension of the classical Arbitrage Pricing Theory (APT) models [18,8]: every asset price process is driven by a source of randomness, which is common to all assets and represents the systematic risk in the market, and by a specific (or idiosyncratic) source of randomness, which affects only that particular asset. They obtain a *well diversified portfolio* as the limit of a sequence of portfolios based on the first n assets: this is evidently an idealized picture of a financial market where we are allowed to trade on infinitely many assets. The real world counterpart of a *well diversified portfolios* is for instance a mutual fund which can contain hundreds of different assets (whereas usually a small investor can only trade on a very limited number of assets). The main result of Björk and Näslund is that, under some technical conditions, they can build a well diversified portfolio which is driven only by the systematic risk and which completes the small markets.

A more systematic investigation of completeness in large financial markets is the topic of the paper [3], where a diversifying strategy is represented as a process which is integrable with respect to the sequence of the price processes. Furthermore, in this paper, the martingale model is chosen: the asset price processes are martingales, so it sufficient to adapt the cylindrical stochastic integration theory introduced by Mikulevicius and Rozovskii [17] to the case of a sequence of martingales.

The aim of the present paper is to investigate further large financial markets (and in particular the factor models considered by Björk and Näslund) by considering the problems of *super-replication* and *utility maximization*. In order to face these problems the martingale model is no longer appropriate. It is therefore necessary to have a theory of stochastic integration with respect to a sequence of semimartingales. This can be found in [4] and we refer to that paper for all results pertaining to this stochastic integration theory. We call *generalized strategy* (as opposed to elementary strategy) a process which is integrable with respect to the whole semimartingale sequence. This notion formalizes the idea of a portfolio in which each asset can contribute, possibly with an infinitesimal weight.

In Section 2 we describe our model and the class of the admissible strategies. For a generalized strategy, the proper notion of admissibility cannot refer uniquely to the limit (generalized) integrand, but has to take into account also a sequence of approximating elementary strategies: the reason for this is that the so called *Ansel–Stricker theorem* is no more valid in this infinite-dimensional context.

Section 3 gives the main *super-replication* result. It is well-known that in the *small market* based on the first n assets, the super-replication price is given by

$$\pi_n(X) = \sup_{Q \in \mathscr{M}_e^n} E_Q[X],$$

where $\mathscr{M}_e^n$ is the set of the equivalent martingale measures for the first n assets [5,6]. If we assume that there exists at least an equivalent martingale measure for the whole sequence of the price processes (that is, $\bigcap_{n \geqslant 1} \mathscr{M}_e^n \neq \emptyset$), we have an analogous characterization for the super-replication price with the appropriate definition of admissible generalized strategies. Clearly, the sequence of the prices $(\pi_n(X))$ is decreasing: we show with two examples that the limit may be strictly greater than the super-replication price in the large market.

The dual characterization of the super-replication price paves the way to an extension of the convex duality approach in order to study the *utility maximization problem* in a large market, that is the problem to find

$$\max_{H \in \mathscr{A}} E\left[U\left(x + \int_0^T H_t \, \mathrm{d}S_t\right)\right],$$

where U is a utility function and $\mathscr{A}$ the set of admissible generalized strategies. In Section 4, we show that under suitable assumptions on U, the problem of utility maximization has an optimal solution and in some cases we are able to give an explicit characterization of it. Contrary to the case of the super-replication price, we show that the supremum of the expected utilities over all the elementary strategies coincides with the supremum over the generalized strategies.

Finally, Section 5 is devoted to the analysis of the infinite-dimensional factor models considered by Björk and Näslund: we show that in the large market there is an explicit characterization of the solution, while this may be not possible in every n-dimensional small market. Furthermore, it is shown that in the large market an extension of the so called *Merton's mutual fund theorem* is valid, though this result does not hold in every small market.

2. The model

We consider the model of a financial market with countably many assets: we assume, as in [1,3], that there is one fixed market which consists of a riskless asset S^0, used as numéraire, with price constantly equal to 1, and countably many risky assets, which are modeled by a sequence of semimartingales $(\mathbf{S}_t)_{t\in[0,T]} = ((S^i_t)_{t\in[0,T]})_{i=1}^{\infty}$, based on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$, which satisfies the usual assumptions.

We begin our discussion with the following:

Definition 2.1. (i) An *n-elementary strategy* is an $\mathbb{R}^n$-valued, predictable process, integrable with respect to $(S^i)_{i\leqslant n}$. An *elementary strategy* is a strategy which is n-elementary for some n.

(ii) Let $x \in \mathbb{R}^+$: an n-elementary strategy H is said to be *x-admissible*, if $(H \cdot \mathbf{S})_t = \int_0^t \sum_{i\leqslant n} H^i \, \mathrm{d}S^i \geqslant -x$ a.s. An elementary strategy H is called *admissible* if it is x-admissible for some $x \in \mathbb{R}_+$.

We denote by $\mathscr{H}^n$ the set of admissible n-elementary strategies and by $\mathscr{H}$ the set of admissible elementary strategies.

As usual, the notation $H \cdot S$ denotes the stochastic integral process $\int_0^{\cdot} H_t \, \mathrm{d}S_t$: we point out that, although in Definition 2.1(ii) $\mathbf{S}$ denotes an infinite-dimensional semimartingale, the process $(H \cdot \mathbf{S})$ is a standard stochastic integral in $\mathbb{R}^n$.

Essentially, elementary strategies are those involving only a finite number of assets. These strategies should be allowed by any reasonable definition of admissibility, therefore any no-arbitrage condition should exclude elementary arbitrage strategies. Since in finite-dimensional markets the absence of arbitrage is conceptually equivalent to the existence of (local) martingale measures (see [5] for a precise statement), we define the following sets of (local) martingale measures:

$$\mathscr{M}^n = \{Q \ll P \,|\, (H \cdot \mathbf{S}) \text{ is a } Q\text{-local martingale for all } H \in \mathscr{H}^n\},$$
$$\mathscr{M}^n_e = \{Q \in \mathscr{M}^n \,|\, Q \sim P\},$$
$$\mathscr{M} = \bigcap_{n\geqslant 1} \mathscr{M}^n, \quad \mathscr{M}_e = \bigcap_{n\geqslant 1} \mathscr{M}^n_e.$$

Remark 2.1. The above definition of $\mathscr{M}^n$ is given in analogy to [14], and it does not imply that each S^i is a local martingale under any $Q \in \mathscr{M}_e$. However, this property is recovered when S^i is locally bounded (see [5] for details).

By the Fundamental Theorem of Asset Pricing, the absence of elementary arbitrage strategies implies that $\mathscr{M}^n_e$ is nonempty for all n. Throughout this paper, we shall make the stronger assumption:

Assumption 2.1. The set $\mathscr{M}_e$ is not empty.

The class $\mathscr{H}$ of elementary strategies fails to be closed in any reasonable sense, therefore it is not suitable for optimization problems. Indeed, the classical results on APT suggest that optimal policies should involve the use of infinitely many assets to maximize the effects of diversification.

Closing the space of elementary strategies naturally leads to definition of stochastic integrals with respect to a sequence of semimartingales. We recall the main definitions from [4], where such a definition has been introduced. We denote by $E = \mathbb{R}^{\mathbb{N}}$ the set of all real sequences and by E' its topological dual, which is the set of linear combination of Dirac measures on $\mathbb{N}$. We call simple integrand an E'-valued process of the form $H = \sum_{i \leqslant n} h^i \delta_i$, where, as usual, δ_i denotes the Dirac delta at point i, and h^i are bounded and predictable processes. For a simple integrand, $(H \cdot \mathbf{S})$ is defined as the finite-dimensional stochastic integral $(\sum_{i \leqslant n} h^i \cdot S^i)$.

A generalized integrand will be obtained as the limit, in a sense to be made precise, of simple integrands.

First, we recall the definition of unbounded functionals.

Definition 2.2. An *unbounded functional* on E is a linear functional k whose domain $\mathrm{Dom}(k)$ is a subspace of E.

Definition 2.3. (i) A process $\mathbf{H}$ with values in the set of unbounded functionals on E is *predictable* if there exists a sequence (H^n) of simple processes, such that

$$\mathbf{H} = \lim_{n \to \infty} H^n \quad \text{a.s.}$$

this means that $x \in \mathrm{Dom}(\mathbf{H})$ if the sequence $H^n(x)$ converges and $\mathbf{H}(x) = \lim_n H^n(x)$.

(ii) A predictable process $\mathbf{H}$ with values in the set of unbounded functionals on E is *integrable* with respect to $\mathbf{S}$ if there exists a sequence (H^n) of simple integrands such that H^n converges to $\mathbf{H}$ and the sequence of semimartingales $(H^n \cdot \mathbf{S})$ converges to a semimartingale Y in the semimartingale topology (see [7]). In this case, we define $\mathbf{H} \cdot \mathbf{S} = Y$.

Definition 2.3(ii) makes sense if the limit process Y is unique, namely if it is independent of the approximating sequence: this was proved in [4, Proposition 5.1]. We refer to [4] for all the properties of this stochastic integral. In particular, it can be shown that the integral $\mathbf{H} \cdot \mathbf{S}$ is linear with respect to $\mathbf{H}$, but not with respect to $\mathbf{S}$. The most relevant result in [4] is Theorem 5.2, which is the extension of a result originally proved by Mémin [16] in the finite-dimensional context: the limit of a sequence of stochastic integrals is still a stochastic integral, possibly with a generalized integrand. This result will be used in the next section. In light of these results, we give the following definition:

Definition 2.4. A *generalized strategy* is a process $\mathbf{H}$ which is integrable with respect to the semimartingale $\mathbf{S}$.

Given a generalized strategy $\mathbf{H}$, we call *self-financing portfolio* with initial endowment x, the process whose value is defined by the formula:

$$V_t = x + \int_0^t \mathbf{H}_s \, \mathrm{d}\mathbf{S}_s. \tag{2.1}$$

Essentially, V is the limit of the sequence of self-financing portfolios $V^n = x + H^n \cdot \mathbf{S}$, where (H^n) is an approximating sequence for $\mathbf{H}$. This definition, which seems the natural extension to the infinite-dimensional framework of the classical notion of self-financing portfolios, deserves however some comments.

In general, a *generalized strategy* is not necessarily integrable with respect to the sequence of the non-discounted assets. Indeed, assume that the numéraire S^0 has the form $S_t^0 = e^{rt}$ with r constant. Denote by $\mathbf{S}$ the sequence of the non-discounted assets, by $\widetilde{\mathbf{S}}$ that of the discounted assets $\widetilde{S}^i = S^i/S^0$ and by $\mathbf{H}$ a generalized strategy, that is, by definition, a process with values in the set of unbounded functionals on E, which is integrable with respect to $\widetilde{\mathbf{S}}$.

By Ito's formula we have $\mathrm{d}\widetilde{S}_t^i = -rS_t^i \mathrm{e}^{-rt}\,\mathrm{d}t + \mathrm{e}^{-rt}\,\mathrm{d}S_t^i$; hence, if $\mathbf{H}$ is integrable with respect to $\widetilde{\mathbf{S}}$ and $(H^n) = (h^{n,i})_{i\leqslant n}$ is an approximating sequence of simple integrands, a necessary condition for $\mathbf{H}$ to be also $\mathbf{S}$-integrable is that the sequence $H^n(\mathbf{S}) = \sum_{i\leqslant n} h^{n,i} S^i$ converges, that is $\mathbf{S} \in \mathrm{Dom}(\mathbf{H})$ (or, equivalently, $\widetilde{\mathbf{S}} \in \mathrm{Dom}(\mathbf{H})$).

Moreover, if this condition does not hold, it is not possible to determine the amount invested in the riskless bond.

Indeed, let us consider the self-financing portfolio generated by the strategy $\mathbf{H}$ with initial value V_0. Then, $V = V_0 + \mathbf{H}\cdot\widetilde{\mathbf{S}}$ is the limit in $\mathscr{S}(\mathbb{P})$ of the sequence $V^n = V_0 + H^n\cdot\widetilde{\mathbf{S}}$. For all n, we can define the amount invested in the money market account as $\alpha^{0,n} = V_0 + H^n\cdot\widetilde{\mathbf{S}} - H^n(\widetilde{\mathbf{S}})$. However, if $\widetilde{\mathbf{S}} \notin Dom(\mathbf{H})$, hence, if $H^n(\widetilde{\mathbf{S}})$ does not converge, then the sequence $\alpha^{0,n}$ does not converge as well, hence α^0 is not defined. In conclusion, the process (2.1) exists as the limit value of a sequence of self-financing portfolios, but it may be not possible to specify which proportion of the portfolio is based on the riskless bond (see also [3] for an analogous discussion). However, we must not forget that, in the real world, every portfolio is in fact based on a finite (though possibly very large) number of assets.

A drawback of generalized integrands is that the so-called Ansel-Stricker property does not hold: more precisely, if each S^i is a local martingale, $\mathbf{H}$ is a generalized integrand and the process $\mathbf{H}\cdot\mathbf{S}$ is bounded from below, then $\mathbf{H}\cdot\mathbf{S}$ is not necessarily a supermartingale. A counterexample can be found in [4] (Example 5.1), which is an extension of an example due to Emery. Conversely, if H^n is an approximating sequence for $\mathbf{H}$ such that the sequence $H^n\cdot\mathbf{S}$ is uniformly bounded from below and converges to $\mathbf{H}\cdot\mathbf{S}$, then it is easy to verify that $\mathbf{H}\cdot\mathbf{S}$ is a supermartingale. Therefore, a good definition of admissibility has to take into account also an approximating sequence:

Definition 2.5. Let $x>0$. A strategy $\mathbf{H}$ is *x-admissible* if there exists an approximating sequence $(H^n)_{n\geqslant 1} \subset \mathscr{H}^n$ of x-admissible strategies, such that $(H^n\cdot\mathbf{S}) \to (\mathbf{H}\cdot\mathbf{S})$ in the semimartingale topology. We denote the set of x-admissible generalized strategies by $\mathscr{A}_x$.

In practice, the set A_x contains those strategies which can be approximated by elementary strategies, each of them admissible with the same capital x.

In the next section we show that the class $\mathscr{A}_x$ is a good definition of admissible claims, in the sense that the following properties hold:

(i) Assumption 2.1 excludes arbitrage opportunities.
(ii) Claims dominated by a fixed capital admit a dual characterization.
(iii) The maximum expected utility on the entire market is the limit of maximum expected utility on finite-dimensional submarkets.

3. Dual characterization of superreplicable claims

In finite-dimensional markets, we have that, for any $X \in L^0_+$ and $x > 0$:

$$\sup_{Q \in \mathscr{M}^n_e} E_Q[X] \leqslant x \iff X \leqslant x + (H \cdot \mathbf{S})_T \quad \text{for some } H \in \mathscr{H}^n. \tag{3.1}$$

This characterization of contingent claims superreplicable with initial capital x was first proved in [6] in the Brownian filtration, and later extended to the general case by a number of authors (see [5] and the references therein).

We denote by $\pi_n(X)$ the superreplication price of X using the first n securities:

$$\pi_n(X) = \sup_{Q \in \mathscr{M}^n_e} E_Q[X].$$

As we consider the entire market, we have two possible analogues for the left-hand side in (3.1):

$$\pi_\infty(X) = \lim_{n \to \infty} \pi_n(X) = \inf_{n \geqslant 1} \sup_{Q \in \mathscr{M}^n_e} E_Q[X],$$

$$\pi(X) = \sup_{Q \in \mathscr{M}_e} E_Q[X].$$

It is clear from the definition that the following inequality holds:

$$\pi(X) \leqslant \pi_\infty(X).$$

We will show with some examples (see Example 3.1 and Section 5) that the strict inequality may hold. The number $\pi_\infty(X)$ can be easily characterized as follows:

$$\pi_\infty(X) = \inf\{x \mid X \leqslant x + (H \cdot \mathbf{S})_T, \text{ for some } H \in \mathscr{H}\}.$$

In general $\pi_\infty(X)$ is only an infimum.

The next theorem provides a characterization of $\pi(X)$ analogous to (3.1) in terms of generalized admissible strategies.

Theorem 3.1. *Let $X \in L^0_+$ and $x > 0$. The following conditions are equivalent*:

(i) $\sup_{Q \in \mathscr{M}_e} E_Q[X] \leqslant x$;

(ii) *There exists $\mathbf{H} \in \mathscr{A}_x$, such that*

$$X \leqslant x + (\mathbf{H} \cdot \mathbf{S})_T.$$

In order to prove Theorem 3.1, we need to introduce some notation. For a set $A \subset L^0$, we denote by $\widehat{A}$ its closure in the space L^0, endowed with convergence in probability. Following Kramkov and Schachermayer [14], we introduce the sets

$$\mathscr{C}_n = \{X \in L^0_+ : X \leqslant 1 + (H \cdot \mathbf{S})_T, H \in \mathscr{H}^n\},$$

and $\mathscr{C} = \bigcup_{n \geqslant 1} \mathscr{C}_n$. Recall that the polar of a set $A \subset L^0_+$ is defined by:

$$A^\circ = \{f \in L^0_+ : E[fg] \leqslant 1 \text{ for all } g \in A\}.$$

We need a few preliminary results:

Lemma 3.2. *Let $X \in L^0_+$. Then, under Assumption 2.1 we have that*

$$\sup_{Q \in \mathscr{M}_e} E_Q[X] = \sup_{f \in \mathscr{C}^\circ} E[fX].$$

Proof. Note first that, if we identify the elements of $\mathscr{M}_e$ with their Radon Nykodim derivatives, $\mathscr{M}_e \subset \mathscr{C}^\circ$. In fact, for any admissible $H \in \mathscr{H}$, we have that $(H \cdot \mathbf{S})$ is a supermartingale, and hence $E_Q[1 + (H \cdot \mathbf{S})_T] \leqslant 1$.

Vice versa, suppose that $f \in \mathscr{C}^\circ$. Since $1 \in \mathscr{C}$ and $-L^\infty_+ \subset \mathscr{C}$, it follows that $f \in L^1_+$ and that $E[f] \leqslant 1$. Let $(f_n)_{n=1}^\infty$ be a maximizing sequence for the right-hand side. Up to a rescaling, which can only increase expectations, we can assume that $E[f_n] = 1$ for all n. By assumption 2.1, we can consider $Q \in \mathscr{M}_e$, and denote by $g = \frac{\mathrm{d}Q}{\mathrm{d}P}$ its density. We define a new sequence of measures $(Q_n)_{n=1}^\infty$, defined by $\frac{\mathrm{d}Q_n}{\mathrm{d}P} = (1 - \frac{1}{n}) f_n + \frac{1}{n} g$. It is clear that $Q_n \in \mathscr{M}_e$ for all n, and that $\lim_{n\to\infty} E_{Q_n}[X] = \lim_{n \to \infty} E[f_n X]$. $\square$

Lemma 3.3. *Let $(f_n)_{n=1}^\infty$ be a sequence of random variables, such that $-1 \leqslant f_n \leqslant (H^n \cdot \mathbf{S})_T$, where $H^n \in \mathscr{H}$. Assume that f_n converges almost surely to f. Then, there exists a process $\mathbf{H} \in \mathscr{A}_1$ such that $f \leqslant (\mathbf{H} \cdot \mathbf{S})_T$.*

Proof. The proof essentially follows from the results in Section 4 in [5] (see also [9]). Assumption 2.1 immediately implies that $(H^n \cdot \mathbf{S})_t \geqslant -1$ for all $t \leqslant T$, which means that H^n is 1-admissible. Let us denote by $K^1_0 = \{(H \cdot \mathbf{S})_T : H \in \mathscr{H}, H \text{ 1-admissible}\}$. By Lemma A1.1 in [5], there exists a sequence of convex combinations $(\tilde{H}^n) \in \mathrm{conv}(H^n, H^{n+1}, \ldots)$, such that $(\tilde{H}^n \cdot \mathbf{S})_T$ converges almost surely. By the convexity of K^1_0, $\tilde{H}^n$ are still 1-admissible elementary strategies. It follows that $f \leqslant g$, where g is some element in $\widehat{K}^1_0$, hence the set $\mathscr{D}_f = \{g \in \widehat{K}^1_0 :\ g \geqslant f \text{ a.s.}\}$ is not empty. Since Assumption 2.1 implies that K^1_0 is bounded in L^0, it follows that $\mathscr{D}$ is also bounded. Lemma 4.3 in [5] implies that $\mathscr{D}_f$ contains a maximal element, denoted by f_0, which can be written in the form $f_0 = \lim_n (L^n \cdot \mathbf{S})_T$, where L^n are 1-admissible elementary strategies and the convergence is in probability. Then we can apply Lemmas 4.5, 4.10 and 4.11 in [5], and we obtain a sequence of strategies $(\tilde{L}^n) \in \mathrm{conv}(L^n, L^{n+1}, \ldots)$ such that the sequence of semimartingales $(\tilde{L}^n \cdot \mathbf{S})$ is Cauchy in the semimartingale topology.

At this point, if we were in the finite-dimensional setting (and dealed with standard stochastic integrals in $\mathbb{R}^d$, for $d \in \mathbb{N}$), we would apply a result due to Mémin (Corollary III.4 in [16]) and claim that $(\tilde{L}^n \cdot S)$ converges to a stochastic integral $L \cdot S$. In the present case, we can apply Theorem 5.2 in [4] (which may be seen as an infinite-dimensional analogous of the result of Mémin), to obtain a generalized strategy $\mathbf{H}$ such that $\tilde{L}^n$ converges to $\mathbf{H}$ and $(\tilde{L}^n \cdot \mathbf{S})$ converges to $\mathbf{H} \cdot \mathbf{S}$ in the semimartingale topology, hence $\mathbf{H} \in \mathscr{A}_1$. $\square$

The previous lemma allows to characterize the closure of $\mathscr{C}$.

Lemma 3.4. *The following result holds*:

$$\widehat{\mathscr{C}} = \{X \in L^0_+ : X \leqslant 1 + (\mathbf{H}\cdot\mathbf{S})_T,\ \mathbf{H} \in \mathscr{A}_1\}.$$

Proof. Let $(X_n)_{n=1}^{\infty}$ be a sequence in $\mathscr{C}$, converging in probability to a random variable X. Up to a subsequence, we can assume that X^n converges almost surely to X. Then, Lemma 3.3 applied to the sequence $(X^n - 1)$ shows that there exists a generalized strategy $\mathbf{H} \in \mathscr{A}_1$ such that $X \leqslant 1 + (\mathbf{H}\cdot\mathbf{S})_T$.

Conversely, assume that $X \leqslant Y = 1 + (\mathbf{H}\cdot\mathbf{S})_T$ for some $\mathbf{H} \in \mathscr{A}_1$. The random variable Y belongs to $\widehat{\mathscr{C}}$, which is solid (we recall that a subset $A \subset L^0_+$ is called solid if $g \in A$, $h \in L^0$ and $0 \leqslant h \leqslant g$ implies that $h \in A$). It follows that $X \in \widehat{\mathscr{C}}$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We can assume, without loss of generality, that $x = 1$. By Lemma 3.2, condition (i) amounts to say that X belongs to $\mathscr{C}^{\circ\circ}$, which is the bipolar of $\mathscr{C}$. By the bipolar theorem (in the version of Brannath and Schachermayer [2]), $\mathscr{C}^{\circ\circ}$ is the closed convex solid hull of $\mathscr{C}$ in L^0_+. Since $\mathscr{C}$ is convex and solid, $\mathscr{C}^{\circ\circ}$ is just the closure of $\mathscr{C}$ in L^0. Then, the equivalence between (i) and (ii) follows from the characterization of $\widehat{\mathscr{C}}$ given in Lemma 3.4. □

The next example shows that, with infinitely many assets, the superreplication prices $\pi(X)$ and $\pi_\infty(X)$ may be different.

Example 3.1. Consider a one-period model (i.e. $T = \{0, 1\}$) on the countable probability space $\Omega = \{\omega_n\}_{n=0}^{\infty}$. $\mathscr{F}_0$ and $\mathscr{F}_1$ are, respectively, the trivial and the discrete σ-algebra on Ω. The probability and the assets prices at the final time 1 are defined as follows:

$$\begin{aligned} &P(\omega_0) = 1 - \alpha, \quad P(\omega_n) = \alpha 2^{-n} \quad \text{for } n \geqslant 1, \\ &S^n_1(\omega_0) = 1, \quad S^n_1(\omega_n) = 2^n, \quad S^n_1(\omega_k) = 0 \quad \text{for all } k \notin \{0, n\}. \end{aligned}$$

We set the initial price of all assets to some constant $c > 0$. Note that this market is complete: to see this, it is sufficient to show that all Arrow–Debreu securities $X_j : \omega_i \mapsto \delta_{ij}$ are replicable. For $k \geqslant 2$, we have trivially $X^k = 2^{-k} S^k$, hence it is sufficient to replicate X^0 (X^1 will be obtained from the riskless asset by difference). Consider the strategy of borrowing one unit of the riskless asset, and holding $\{\theta_n\}_{n=1}^{\infty}$ units of risky assets. If we set $\theta_1 = 1$ and $\theta_n = 2^{-n}$, the payoff of this strategy will be exactly X^0.

Let us now consider the cost of superreplicating the claim X^0. If we have only a finite number of assets at our disposal, it is intuitively clear that this cost will be at least c. This can be seen as follows: let Q be a martingale measure for all S^n, and denote by $q_n = Q(\omega_n)$. In the market with the first n securities, we have the system of

n equations in $n+1$ unknowns:

$$\begin{aligned} q_0 + q_1 &= c \\ q_0 + 2q_2 &= c \\ &\vdots \\ q_0 + 2^n q_n &= c, \end{aligned}$$

which has one-dimensional set of solutions:

$$\begin{aligned} &q_0 \in (0, c), \\ &q_k = 2^{-k}(c - q_0), \quad 1 \leqslant k \leqslant n \end{aligned}$$

hence the supremum of q_0 (the price of X^0 under Q) is clearly c.

Note that for finite n the condition $\sum_{k=1}^{\infty} q_k = 1$ remains vacuous, but this is no longer true when $n = \infty$. In this case, the only martingale measure Q is given by:

$$\begin{aligned} &q_0 = 2c - 1, \\ &q_k = 2^{-k}(c - q_0), \quad k \geqslant 1, \end{aligned}$$

and $2c - 1 < c$ whenever $c < 1$.

4. Utility maximization

In this section we want to study the problem of utility maximization, using the convex duality approach.

With the following definition, we summarize the usual assumptions on utility functions.

Definition 4.1. A function $u : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfies the *Regularity Conditions* if it is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$.

Given a utility function $U : \mathbb{R}_+ \mapsto \mathbb{R}$, we make the following assumption:

Assumption 4.1. The utility function U satisfies the *Regularity Conditions*.

As usual, V will denote the convex conjugate function of U, namely

$$V(y) = \sup_{x>0} [U(x) - xy]$$

for $y > 0$. It is well-known (see, for instance, [20]) that, if U satisfies the regularity conditions, then V satisfies the inversion formula

$$U(x) = \inf_{y>0} [V(y) + xy].$$

In addition, V is continuously differentiable, strictly convex, strictly decreasing, $V'(0) = -\infty$, $V'(\infty) = 0$; in other words $(-V)$ satisfies the *Regularity Conditions*.

Finally, $-V'(y)$ is the inverse function of $U'(x)$ (for more details and further references, one can see [19]).

For all $n \geqslant 1$, we define the finite-dimensional value functions:

$$u_n(x) = \sup_{H \in \mathscr{H}^n} \mathbb{E}\left[U\left(x + \int_0^T H_s \, d\mathbf{S}_s\right)\right] = \sup_{X \in x\mathscr{C}_n} \mathbb{E}[U(X)]. \tag{4.1}$$

We denote by $\mathscr{D}_n$ the polar of $\mathscr{C}_n$: this set was characterized by Kramkov and Schachermayer as the closed, convex, solid hull of the set $\mathscr{M}_e^n$ (see [14] for details). The dual problem of (4.1) is then defined by

$$v_n(y) = \inf_{Y \in \mathscr{D}_n} \mathbb{E}[V(yY)]. \tag{4.2}$$

If we assume that $u_n(x) < \infty$ for all x, the function $v_n(y)$ is the convex conjugate of $u_n(x)$ [14, Theorem 2.1]. Let us define

$$u_\infty(x) = \lim_n u_n(x), \quad v_\infty(y) = \lim_n v_n(y);$$

clearly, $u_\infty(x) = \sup_{H \in \mathscr{H}} \mathbb{E}[U(x + \int_0^T H_s \, d\mathbf{S}_s)]$, that is, $u_\infty(x)$ is the value function of the utility maximization problem over all the elementary strategies. To exclude trivial cases, we assume that $u_\infty(x_0) < \infty$ for some $x_0 > 0$ (equivalently, for all $x > 0$, by the concavity of u_∞).

We now consider the problem of maximizing expected utility over the class of generalized strategies $\mathscr{A}$:

$$\max_{\mathbf{H} \in \mathscr{A}_x} E[U(x + (\mathbf{H} \cdot \mathbf{S})_T)] \tag{4.3}$$

and its value function:

$$u(x) = \sup_{\mathbf{H} \in \mathscr{A}_x} E[U(x + (\mathbf{H} \cdot \mathbf{S})_T)].$$

Since $\mathscr{C} = \bigcup_{n \geqslant 1} \mathscr{C}_n$, it is easy to check that $\mathscr{C}^\circ = \bigcap_{n \geqslant 1} \mathscr{D}_n = \mathscr{D}$: then, the dual value function is defined as follows:

$$v(y) = \inf_{f \in \mathscr{D}} E[V(yf)]. \tag{4.4}$$

The inequalities $u(x) \geqslant u_\infty(x)$ and $v(y) \geqslant v_\infty(y)$ are evident: in fact, we will prove that equality holds in both cases. Let us start by proving the second one:

Lemma 4.1. $v(y) = v_\infty(y)$, *for all* $y > 0$.

Proof. Let $Y_n \in \mathscr{D}_n$ be such that

$$\lim_{n \to \infty} E[V(yY_n)] = v_\infty(y).$$

Since $1 \in \mathscr{C}_n$ for all n, $(Y_n)_{n=1}^\infty$ is bounded in $L^1(Q)$ for any $Q \in \mathscr{M}_e$, and *a fortiori*, it is bounded in L^0. By Lemma A1.1 in [5], there exists a sequence

$(Z_n) \in \operatorname{conv}(Y_n, Y_{n+1}, \ldots)$, which converges almost surely to a random variable $Z \in L^0$. Also, since $Z_n \in \mathcal{D}_n$, it follows that $Z = \lim_{n\to\infty} Z_n \in \bigcap_{n=1}^{\infty} \mathcal{D}_n = \mathcal{D}$. By the convexity of V, it is easily seen that:

$$\lim_{n\to\infty} E[V(yZ_n)] = v_\infty(y).$$

Finally, Lemma 3.4 in [14] implies that the sequence $(V^-(yZ_n))$ is uniformly integrable (since (Z_n) is bounded in $L^1(P)$), hence

$$v(y) \leqslant E[V(yZ)] \leqslant \liminf_{n\to\infty} E[V(yZ_n)] = v_\infty(y). \qquad \square$$

Lemma 4.2. *There exists y_0 such that $v(y) < \infty$ for $y > y_0$.*

Proof. For all n, the following relation holds for $y > 0$ (see [14, Theorem 3.1]):

$$v_n(y) = \sup_{x>0}(u_n(x) - xy)$$

and hence

$$v(y) \leqslant \sup_{x>0}(u_\infty(x) - xy). \tag{4.5}$$

Since u_∞ is concave, the thesis easily follows. $\square$

Proposition 4.3. $u_\infty(x) = u(x)$, *for all $x > 0$.*

Proof. Let $X \in x\widehat{\mathcal{C}}$, $Y \in y\mathcal{D}$. Since $U(X) \leqslant V(Y) + XY$, it follows that $u(x) \leqslant v(y) + xy$ for all $y > 0$, and therefore

$$u(x) \leqslant \inf_{y>0}(v(y) + xy).$$

In particular, it follows that $u(x) < \infty$ for all $x > 0$. We can then apply Theorem 3.1 in [14], to prove that u and v are in duality and v is the convex conjugate of u: precisely,

$$u(x) = \inf_{y>0}(v(y) + xy),$$

$$v(y) = \sup_{x>0}(u(x) - xy).$$

Denote by $\tilde{v}$ the convex conjugate of u_∞. Since (4.5) holds, we have that $\tilde{v}(y) \geqslant v(y)$ for all y. So, we obtain $u_\infty(x) \geqslant u(x)$, which completes the proof. $\square$

The next theorem resumes the main results on the utility maximization problems, which follow from [15] and from the results proved above.

Theorem 4.4. *Under Assumptions* 2.1 *and* 4.1, *we have that*:

(i) *The value functions u and v are conjugate*; *moreover*, *u and $-v$ satisfy the Regularity Conditions.*

(ii) *The function v satisfies the representation*:

$$v(y) = \inf_{Q \in \mathscr{M}_e} E\left[V\left(y\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

(iii) *The following relation holds*:

$$u(x) = \sup_{H \in \mathscr{H}} E[U(x + (H \cdot \mathbf{S})_T)].$$

Furthermore, *if* $v(y) < \infty$ *for all* $y > 0$ (*see* [15]), *then*

(iv) *The optimal solution* $\hat{X}(x) = (\hat{\mathbf{H}}(x) \cdot \mathbf{S})_T$ *to* (4.3) *exists for any* $x > 0$, *and* $\hat{X}(x)$ *is unique. In addition*, *if* $y = u'(x)$, *we have that* $U'(\hat{X}_T(x)) = \hat{Y}_T(y)$, *where* $\hat{Y}(y)$ *is the optimal solution to* (4.4).

5. Factor models

In this section, we want to analyze a model introduced by Björk and Näslund [1], as a continuous time extension of the classical Arbitrage Pricing Models studied in [18,8]. We assume that every asset price depends on a systematic source of randomness which affects all the assets and on an idiosyncratic source of randomness which is typical for that asset. In particular, we assume that the price processes evolve according to the following dynamics:

$$\mathrm{d}S_t^i = S_{t-}^i(\alpha_i\,\mathrm{d}t + \beta_i\,\mathrm{d}\hat{N}_t + \sigma_i\,\mathrm{d}W_t^i),$$

where $(W^i)_{i \geqslant 1}$ is a sequence of independent Wiener processes and $\hat{N}_t = N_t - \lambda t$ is a compensated Poisson process with intensity λ (N is the Poisson process), independent of W^i for all i. The Poisson process models some shocks which may occur in the market and affect all the assets. As in [1], the coefficients $\alpha_i, \beta_i, \sigma_i$ are uniformly bounded deterministic constants. In particular we assume that $\beta_i,\ \sigma_i \geqslant \varepsilon > 0$ for all i and there exists M such that $\sup_i(|\alpha_i|, \beta_i, \sigma_i) \leqslant M$. Björk and Näslund studied the question of No Arbitrage and completeness and showed that an asymptotic portfolio can be defined, as a limit of well-diversified portfolios, in order to complete the market. In [3], completeness was characterized under an equivalent probability measure. Here, we want to analyze the problems of superreplication and utility maximization.

We assume for simplicity that $T = 1$ and take as filtration $(\mathscr{F}_t)_{t \leqslant 1}$ the filtration generated by the price processes, hence by $\{(W^i)_{i \geqslant 1}, N\}$. Every local martingale L has necessarily the form

$$L_t = L_0 + \int_0^t k_s\,\mathrm{d}\hat{N}_s + \sum_{i \geqslant 1} \int_0^t h_s^i\,\mathrm{d}W_s^i, \tag{5.1}$$

where $k, (h^i)_{i\geqslant 1}$ are predictable processes and

$$\int_0^1 |k_s|\, \mathrm{d}s + \sum_{i\geqslant 1} \int_0^1 (h_s^i)^2\, \mathrm{d}s < \infty \quad \text{a.s.} \tag{5.2}$$

This fact is a straightforward extension of the analogous (and well-known) result in finite dimension and the proof goes along the same lines: we give a sketch of the proof since it seems difficult to find a precise reference in the literature. Since the integral is an isometry, the space of the stochastic integrals is a closed subspace in L^2; then, it is sufficient to prove that if a random variable Z in $L^2(\mathscr{F}_T)$ is orthogonal to all the stochastic integrals, then it is identically zero. This is obvious (as a consequence of the finite-dimensional results) if Z is measurable with respect to the σ-algebra generated by $W_1, \ldots, W_n$, N, for every $n\geqslant 1$. Since the set of the random variables which fulfill these conditions is a dense subset of $L^2(\mathscr{F}_T)$, the claim follows.

Let Q be a probability measure equivalent to P. Then, its density has the form $\mathrm{d}Q/\mathrm{d}P = \mathscr{E}(L_1)$ (we recall that $\mathscr{E}$ denotes the stochastic exponential), where L has the form (5.1), with $L_0 = 0$; furthermore, $k_s > -1$ to ensure that $\mathscr{E}(L_1) > 0$ and L is such that $\mathscr{E}(L_t)$ is a uniformly integrable martingale.

By Girsanov's theorem, it follows that the process $\tilde{W}_t^i = W_t^i - \int_0^t h_s^i\, \mathrm{d}s$ is a Q-Wiener process, while the process $\tilde{N}_t = \hat{N}_t - \int_0^t \lambda k_s\, \mathrm{d}s = N_t - \int_0^t \lambda(1 + k_s)\, \mathrm{d}s$ is a Q-martingale (namely $\int_0^t \lambda(1 + k_s)\, \mathrm{d}s$ is the Q-compensator of the point process N).

Since $(S^i)_{i\leqslant n}$ is locally bounded, we have that $Q \in \mathscr{M}_e^n$ if and only if $(S^i)_{i\leqslant n}$ is a Q-local martingale and this occurs if and only if

$$h_t^i = -\frac{\alpha_i + \beta_i \lambda k_t}{\sigma_i} \tag{5.3}$$

for all $i\leqslant n$. A necessary condition for Assumption 2.1 to hold is that the above equality is satisfied for all $i\geqslant 1$. Then, by condition (5.2), it must be $\int_0^1 \sum_i (\alpha_i + \beta_i \lambda k_t)^2 \sigma_i^{-2}\, \mathrm{d}t < \infty$, as was shown also by Björk and Näslund. They also showed that the sequence (α_i/β_i) converges to some real number h_0. This implies that $k_t \equiv -h_0/\lambda$, $h_t^i \equiv -(\alpha_i + \beta_i h_0)/\sigma_i$ and there exists a unique equivalent martingale measure Q, provided that $h_0 < \lambda$ (the uniform integrability of the density $\mathscr{E}(L_1)$ is a consequence of Novikov condition).

Conversely, on the n-dimensional market, there are infinitely many equivalent martingale measures. In particular, the point process N may have any intensity, and, possibly, even a stochastic compensator.

5.1. Super-replication price

In Section 3, we observed that we can give two different definitions for the superreplication price in the large market: we can show in this framework an example where the superreplication prices π and π_∞ are different. Assume that $h_0 = 0$ and let $X = \mathbb{1}_{\{N_1=0\}}$ be the binary claim which pays 1 if the market does not jump, 0 otherwise. In the large market, N_1 is a Poisson random variable with intensity λ, hence $\pi(X) = E_Q[X] = e^{-\lambda}$. In the n-market, N_1 may be a Poisson random variable with any intensity (or, possibly, a random variable with more general distribution): it is evident, then, that $\pi_\infty(X) = 1 > e^{-\lambda}$.

5.2. Utility maximization

Let U be a utility function and V its convex conjugate, such that the hypotheses of Section 4 are satisfied. Thanks to the uniqueness of the martingale measure, we have that

$$v(y) = \mathbb{E}\left[V\left(y\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right] = \mathbb{E}[V(y\hat{Y})],$$

where we recall that

$$\begin{aligned}\hat{Y} = \mathscr{E}(L_1) &= \mathscr{E}\left(\sum_{j\geqslant 1} h_j W_1^j - \frac{h_0}{\lambda}\hat{N}_1\right)\\ &= \exp\left(\sum_{i\geqslant 0} h_i^2\right)\mathscr{E}\left(\sum_{j\geqslant 1} h_j \tilde{W}_1^j - \frac{h_0}{\lambda}\tilde{N}_1\right).\end{aligned}$$

Denote by $\tilde{W}_h$ the process $\sum_{j\geqslant 1} h_j \tilde{W}^j$. This is a brownian motion with respect to the probability Q as well as the process $\tilde{N}$ is a Q-compensated Poisson process (with compensator $\lambda(1 - h_0/\lambda)t = (\lambda - h_0)t$). Furthermore, both $\tilde{W}_h$ and $\tilde{N}$ coincide with the values of two self-financing portfolios. To show this, we need to find a pair of generalized strategies $\mathbf{H}_1$ and $\mathbf{H}_2$ such that

$$\tilde{W}_h = \mathbf{H}_1 \cdot \mathbf{S}, \quad \tilde{N} = \mathbf{H}_2 \cdot \mathbf{S}. \tag{5.4}$$

Because the integral is invariant with respect to a change in probability, it is sufficient to find two approximating sequences H_1^n, H_2^n such that H_1^n converges to $\mathbf{H}_1$, H_2^n converges to $\mathbf{H}_2$ and $H_1^n \cdot \mathbf{S}$, $H_2^n \cdot \mathbf{S}$ converge, respectively, to $\tilde{W}_h$ and $\tilde{N}$ in $\mathscr{S}(Q)$. In fact, since S^i is a Q-martingale, the convergence will be in $\mathscr{M}^2_{loc}(Q)$. It was already proved in [3] (Section 4) that such sequences do exist, so we will omit the details.

Observe that $\tilde{W}_h$ and $\tilde{N}$ can be interpreted as mutual funds, composed of a small part of each asset. In particular $\tilde{W}_h$ does not depend on the systematic risk and contain a small part of all the idiosyncratic risks, while $\tilde{N}$ is based only on the systematic risk.

The optimal solution of the problem of utility maximization is given by

$$\hat{X}_1(x) = I(y\hat{Y}) = I\left(y\exp\left(\sum_{i\geqslant 0} h_i^2\right)\mathscr{E}\left((\tilde{W}_h)_1 - \frac{h_0}{\lambda}\tilde{N}_1\right)\right).$$

Let $\hat{X}_t$ be defined as $\hat{X}_t = E_Q[X_1|\mathscr{F}_t]$; the process $(\hat{X}_t)_{t\leqslant 1}$ is a Q-martingale. Furthermore, it is a Q-martingale also with respect to the filtration generated by $(\tilde{W}_h, \tilde{N})$, hence it admits a representation as

$$\hat{X}_1(x) = x + \int_0^1 \phi_s(x)\,\mathrm{d}(\tilde{W}_h)_s + \int_0^1 \psi_s(x)\,\mathrm{d}\tilde{N}_s. \tag{5.5}$$

This, combined with (5.4), allows us to find the optimal strategy $\hat{\mathbf{H}}(x) = \phi(x)\mathbf{H}_1 + \psi(x)\mathbf{H}_2$. Note that $\mathbf{H}_1$ and $\mathbf{H}_2$ depend only on the density of the equivalent martingale measure, while $\phi(x)$ and $\psi(x)$ are the sole processes affected by the choice of the utility function. So, we can claim a sort of mutual fund theorem:

Theorem 5.1. *For any utility function U, the optimal portfolio consists of an allocation between the risk free asset, the mutual fund $\tilde{W}_h$ and the mutual fund $\tilde{N}$.*

Note that in the case considered in Section (5.1), that is $h_0 = 0$, the optimal portfolio is based only on the risk-free asset and on the mutual fund $\tilde{W}_h$. This shows that the utility maximization problem in the large market is quite easy to solve when this is complete. Furthermore, we know that the optimal solution is the limit of the optimal solutions in the finite-dimensional market.

Conversely, if one defines the problem in any finite-dimensional market, one can immediately realize that both the utility maximization problem and its dual problem are very difficult to solve.

The mutual fund theorem relies on the results of completeness of the market and representation of the claims as stochastic integrals (hence, as self-financing portfolios), results which are quite simple to obtain in the market analyzed above. It may be reasonable and interesting to consider more complicated models, where for instance there are more than one common source of randomness (multi-factor models). In this case, further conditions on the coefficients are necessary in order to guarantee the completeness of the market. The research of these types of conditions is beyond the scope of the present paper. However we refer to [1,3], as a guide for further investigations: the first paper follows a more heuristic approach, based on the search for asymptotic assets which complete the market; the second one characterizes completeness in terms of an isomorphism between two appropriate Hilbert spaces. In particular, in the cited papers, it is studied in detail the case where the assets have two systematic risk components (two-factor models), that are a Wiener process and a Poisson process, whereas the idiosyncratic component of risk is given by a sequence of independent Poisson processes (one for each asset): in this framework, necessary and sufficient conditions on the coefficients are given for the market to be complete (Theorem 4.7 in [3]). Following the approaches used in the two papers, one can find (at least sufficient) conditions for the completeness of the market with standard (though cumbersome) calculations.

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