A Remark on the $1/H$-Variation of the Fractional Brownian Motion

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Abstract We give an elementary proof of the following property of H-fractional Brownian motion: almost all sample paths have infinite 1/H-variation on every interval.

Keywords Fractional Brownian motion · p-Variation · Ergodic theorem

1 Introduction and Statement of the Result

Let $(B_t)_{t\geq 0}$ be the *Fractional Brownian Motion* with Hurst (or self-similarity) parameter H, $0 < H < 1$ (we refer for instance to [2] or [5] or [8] p. 273 for the definitions): fix $t > 0$ and let $t_k^n = \frac{kt}{n}$ for n integer and $k = 0, \ldots, n$. It is well known (see e.g. [5] or [9]) that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left| B_{t_{k+1}^n} - B_{t_k^n} \right|^p \overset{L^1(\Omega)}{=} \begin{cases} +\infty & p < 1/H \\ t\, E\left[\left|B_1\right|^{1/H}\right] & p = 1/H \\ 0 & p > 1/H \end{cases} .$$

However, if we define the random variable

$$\mathcal{V}(\omega) = \mathcal{V}_{[0,t]}^{1/H}(\omega) = \sup_{n\,,\,0\leq t_1 < t_2 < \ldots < t_n \leq t} \sum_{i=1}^{n-1} \left| B_{t_{i+1}}(\omega) - B_{t_i}(\omega) \right|^{1/H}$$

then $\mathcal{V}(\omega) = +\infty$ a.s. (note that $\mathcal{V}$ is measurable since the paths of the fractional Brownian motion are continuous and therefore the "sup" can be taken over rationals t_i).

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This property is well known in the case of standard *Brownian Motion* (i.e. $H = 1/2$): it was stated (but without a rigorous proof) by P. Lévy in [6] p. 190, and also the excellent book "Revuz-Yor" quotes the result without a proof (see [8] p. 28). A sketch of the proof (always in the case $H = 1/2$) was given by D. Freedman in [4] p. 48.

For the standard Brownian Motion, there is a more precise (and more technical) result due to Taylor (see [11]): given an increasing function $\psi : [0, +\infty) \to [0, +\infty)$, we can define the ψ–variation of the function f on the interval $[a, b]$ as

$$\sup_{n\,,a=t_1<t_2<\ldots<t_n=b} \psi\big(|f(t_{i+1}) - f(t_i)|\big)$$

(when $\psi(t) = t^p$ it is called the p–variation).

Taylor showed that the *correct* function for the variation of the paths of the BM is the function $\psi_1(s) = s^2/2\log^*\log^* s$ (where $\log^* s = \max(1\,,\,|\log s|)$) in the sense that

$$\mathcal{V}_{\psi^*,[a,b]}(\omega) = \sup_{n\,,a=t_1<t_2<\ldots<t_n=b} \psi\big(|B_{t_{i+1}}(\omega) - B_{t_i}(\omega)|\big)$$

is a finite r.v. but is infinite if ψ_1 is replaced by any function ψ such that $\psi(s)/\psi_1(s) \to +\infty$ as $s \to 0+$.

The impact of the p-variation of the paths for (stochastic) integration is well highlighted by L. Coutin (see [2]) for the case of FBM and by Dudley and Norvaisa (see [3]) for more general stochastic processes.

In the general case of the Fractional Brownian Motion, it is well known that $p = 1/H$ is a *limit* case, and that sample paths are γ-hölder continuous for any $\gamma < H$. The result of Theorem 1 is known since it is a consequence of Theorem IV.5.1 of [1]: their proof, however, is based on a complex technology (the theory of Besov spaces).

The aim of this short note is to give an elementary complete proof suggested by the argument presented in [4]; I want to thank Sara Biagini and Giorgio Letta for a discussion on the subject.

Let us fix H with $0 < H < 1$, let $(B_t)_{t\geq 0}$ be a FBM with parameter H and continuous sample paths: define

$$\mathcal{V}_{[a,b]}(\omega) = \sup_{n\,,a=t_1<t_2<\ldots<t_n=b} \big|B_{t_{i+1}}(\omega) - B_{t_i}(\omega)\big|^{1/H}$$

The statement of the result is as follows

Theorem 1. *There exists a null-set $N \subseteq \Omega$ such that, if $\omega \notin N$, then for every $a < b$, $\mathcal{V}_{[a,b]}(\omega) = +\infty$.*

2 The Proof

In the sequel, λ is the Lebesgue measure on $\mathbb{R}^+$ and p^* is a shorthand for $1/H$. Let U be a finite union of disjoint open intervals $]s_i, t_i[$ of $\mathbb{R}^+$ with $s_i, t_i \in \mathbb{Q}$ and let $\mathcal{U}$ be the collection of such subsets of $\mathbb{R}^+$.

For $U = \cup_{i=1}^n]s_i, t_i[\ (0 \le s_1 < t_1 \le s_2 < \ldots < t_n)$, let $q_U(\omega) = \sum_{i=1}^n |B_{t_i}(\omega) - B_{s_i}(\omega)|^{p^*}$; note that

$$\mathcal{V}_{[a,b]}(\omega) = \sup_{U \in \mathcal{U},\, U \subseteq [a,b]} q_U(\omega)$$

Lemma 1. *Fix $m > 0$ and let $p_m = \mathbf{P}(|B_1|^{p^*} \ge m)$. Let $Z_n = I_{\{|B_n - B_{n-1}|^{p^*} \ge m\}}$: then $M_n = \frac{Z_1 + \cdots + Z_n}{n}$ converges a.s. to $E[Z_1] = p_m$.*

Proof. The sequence of one step increments of B, $X_n = B_n - B_{n-1}$, is stationary, centered Gaussian and with covariance function $R(n) = E[X_1 X_{n+1}]$ which tends to 0 when n goes to infinity (see e.g. [7] p. 274): therefore $(X)_{n \ge 1}$ is ergodic (see [10] p. 413). Now Z_n can be written in the form $Z_n = g(X_n)$ with a borel function g and therefore also $(Z_n)_{n \ge 1}$ is ergodic.

As a consequence of the ergodic theorem (Theorem 3.3 p. 413 of [10]), the sequence of the empirical means $M = (M_n)_{n \ge 1}$ converges a.s. (and in L^1) to $E[Z_1]$.

The key of the proof is the following result:

Lemma 2. *Let $I =]s, t[$ be an open interval with $s, t \in \mathbb{Q}^+$ and fix $m > 0$: let $p_m = \mathbf{P}\{|B_1|^{p^*} \ge m\}$ and $r < p_m$. Then there exists a measurable $A_m \subseteq \Omega$ with $\mathbf{P}(A_m) = 1$ such that for all $\omega \in A_m$ there exists $U_\omega \in \mathcal{U}$ with the properties:*

1. $U_\omega \subset I$
2. $\lambda(U_\omega) > r\, \lambda(I)$
3. $q_{U_\omega}(\omega) \ge m\, \lambda(U_\omega)$

Proof. For $n > 1$ and $i = 0, \ldots, n$ let $t_i^n = s + \frac{i}{n}(t - s)$ and $J_i^n =]t_{i-1}^n, t_i^n[$. Set

$$S_n = \sum_{i=1}^n I_{\left\{|B_{t_{i+1}^n} - B_{t_i^n}|^{p^*} \ge m\, \frac{(t-s)}{n}\right\}}$$

$S_n(\omega)$ counts the number of subintervals J_i^n on which $q_{J_i^n}(\omega) > m\, \lambda(J_i^n)$. Thanks to the self-similarity property of fBm (see e.g. [7] p. 275), S_n is distributed as $Z_n = \sum_{i=1}^n I_{\{|B_{i+1} - B_i|^{p^*} \ge m\}}$.

By the Lemma 1, $\frac{Z_n}{n} \to p_m$ almost surely, whence $\frac{S_n}{n}$ tends to p_m in probability. Modulo a subsequence,

$$\lim_n \frac{S_n}{n} = p_m \text{ a.s.}$$

Call A_m the set on which the above sequence converges: if $\omega \in A_m$ then there exists n_ω such that $\frac{S_n}{n}(\omega) > r$ for $n \ge n_\omega$.

Select $n \geq n_\omega$ and among the subintervals J_i^n exactly those such that $q_{J_i^n}(\omega) \geq m\,\frac{(t-s)}{n}$ and let U_ω be their union. Then

$$\lambda(U_\omega) = S_n(\omega)\frac{t-s}{n} > r\,\lambda(I)$$

and

$$q_{U_\omega}(\omega) \geq S_n(\omega)\,m\frac{(t-s)}{n} = m\,\lambda(U_\omega).$$

The same result holds evidently for an element $U \in \mathcal{U}$. Since $\mathcal{U}$ in countable, we have immediately the following result

Corollary 1. *Fix $m > 0$ and set $r = p_m/2$: there exists a measurable set $C_m \subseteq \Omega$ with $\mathbf{P}(C_m) = 1$ such that, if $\omega \in C_m$ and $V \in \mathcal{U}$, there exists $U_\omega \in \mathcal{U}$ and $U_\omega \subset V$ such that $\lambda(U_\omega) > r\,\lambda(V)$ and $q_{U_\omega}(\omega) \geq m\,\lambda(U_\omega)$.*

Lemma 3. *Fix $0 \leq a < b$; $a, b \in \mathbb{Q}$: then $\mathcal{V}_{[a,b]}(\omega) = +\infty$ a.s.*

Proof. Choose $m > 0$ and apply Lemma 2 to $I =]a,b[$: then for every $\omega \in C_m$ there exists $U_\omega^1 \subset I$ such that $q_{U_\omega^1}(\omega) \geq m\,\lambda(U_\omega^1)$ and $\lambda(U_\omega^1) \geq r\,\lambda(I)$.

Now iterate the procedure, that is apply Corollary 1 to $\big(I \setminus \overline{U}_\omega^1\big)$ $\big(\overline{U}_\omega^1$ is the closure of $U_\omega^1\big)$: thus there exists $U_\omega^2 \subseteq \big(I \setminus \overline{U}_\omega^1\big)$ with $q_{U_\omega^2}(\omega) \geq m\,\lambda(U_\omega^2)$ and $\lambda(U_\omega^2) > r\,\lambda\Big(I \setminus \overline{U}_\omega^1\Big)$.

At the $(k+1)$-th step, we have a subset U_ω^{k+1} of $\big(I \setminus (\overline{U}_\omega^1 \cup \ldots \cup \overline{U}_\omega^k)\big)$ such that $\lambda(U_\omega^{k+1}) > r\,\lambda\big(I \setminus \big(\overline{U}_\omega^1 \cup \ldots \cup \overline{U}_\omega^k\big)\big)$ and $q_{U_\omega^{k+1}}(\omega) \geq m\,\lambda(U_\omega^{k+1})$.

Call $V_\omega^k = U_\omega^1 \cup \ldots \cup U_\omega^k$, then $V_\omega^k \in \mathcal{U}$ and

$$q_{V_\omega^k}(\omega) \geq m\,\lambda(V_\omega^k)$$

Moreover, by induction $\lambda(I \setminus \overline{V}_\omega^k) \leq (1-r)^k(b-a)$ and therefore

$$\sup_k\; q_{V_\omega^k}(\omega) \geq m\,\lim_k \lambda(V_\omega^k) = m\,(b-a)$$

Now, the intersection $C = \bigcap_{m \in \mathbb{N}} C_m$ has probability one and any $\omega \in C$ satisfies

$$\mathcal{V}_{[a,b]}(\omega) \geq m\,(b-a)\;\; \forall m \in \mathbb{N}$$

whence the thesis.

If $N_{[a,b]}$ is the null-set $\big\{\omega \in \Omega \,\big|\, \mathcal{V}_{[a,b]}(\omega) < +\infty\big\}$, then the countable union of all $N_{[a,b]}$ with $a < b$; $a, b \in \mathbb{Q}$, satisfies the hypothesis of Theorem 1.

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