Generalizations of Merton's Mutual Fund Theorem in Infinite-Dimensional Financial Models

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Abstract. This is a review paper, concerning some extensions of the celebrated *Merton's mutual fund theorem* in infinite–dimensional financial models, in particular the so called *Large Financial Markets* (where a sequence of assets is taken into account) and *Bond Markets Models* (where there is a continuum of assets).

In order to obtain these results, an infinite–dimensional Stochastic Integration theory is essential: the paper illustrates briefly a new theory introduced to this extent by M. De Donno and the author.

1. Introduction

The *Mutual Fund Theorem* (also called the *separation theorem*) is a central result in the problem of maximizing the investor's expected utility of the terminal wealth of a portfolio of risky and riskless assets. It states that (under suitable assumptions) the investor's allocation decision can be separated in two steps.

In the first step, an efficient portfolio of risky assets is determined (the mutual fund); and in the second step the investor decides the allocation between this efficient portfolio and the riskless asset. The efficient portfolio is identical for all investors regardless their attitude towards risk, as reflected by their utility functions.

Before introducing the results, let us fix some notations.

We indicate by $\mathbf{S}_t = \big(S_t^0, \ldots, S_t^n\big)_{0\le t\le T}$ the available assets on the market. We suppose that the riskless asset S_t^0 is always equal to 1: this simplifies the exposition, since it avoids the introduction of the riskless interest rate, and is not restrictive (this simply means that we consider discounted prices).

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The risky assets $(S_t^i)_{0\le t\le T}$ are supposed to be *semimartingales* adapted to some filtration $\big(\mathcal{F}_t\big)_{0\le t\le T}$ on a probability space $\big(\Omega,\mathcal{F},\mathbf{P}\big)$.

The portfolio's strategy $\mathbf{H}_t = (H_t^0, H_t^1, \dots, H_t^n)$ is a $(n+1)$–dimensional predictable stochastic process such that the vector stochastic integral $\int_0^t \mathbf{H}_s \mathrm{d}\mathbf{S}_s$ is defined: H_t^i represent the number of assets S^i held at time t and the stochastic integral is the mathematical representation for the gain from trade.

The (discounted) value of the portfolio at time t is the random variable $X_t = \sum_{i=0}^n H_t^i S_t^i$ and the portfolio is said to be *self–financing* if $X_t = X_0 + \int_0^t \mathbf{H}_s \mathrm{d}\mathbf{S}_s$.

An alternative representation of the portfolio's strategy is to consider the $(n+1)$–dimensional stochastic process $\mathbf{u}_t$ where u_t^i is the *proportion* of the capital invested in the asset i. The process $\mathbf{u}_t$ is also called the *relative portfolio.*

One has evidently

$$u_t^i = \frac{H_t^i S_t^i}{X_t} = \frac{H_t^i S_t^i}{\sum_{j=0}^n H_t^j S_t^j}$$

This representation of the strategy is suitable when using control techniques: since $\sum_{i=0}^n u_t^i = 1$, it is convenient to consider $(u_t^1, \dots, u_t^n)$ as a *free* control and consequently $u_t^0 = 1 - \sum_{i=1}^n u_t^i$.

In order to keep the exposition as simple as possible, we restrict ourself to the problem of maximizing the expected utility from terminal wealth (more generally, one can consider the problem of maximizing the utility from consumption and terminal wealth, take into account restrictions on the allowed strategies...).

More precisely, we consider an utility function $U : \mathbb{R} \to [-\infty, +\infty[$, and, given an initial endowment x, the problem is to maximize $\mathbb{E}\big[U(X_T)\big]$ over all possible random variables X_T , where X_T is the value at time T of a self–financing portfolio with $X_0 = x$.

We consider the case where $U(x) = -\infty$ for $x < 0$ (negative wealth is not allowed), and for positive x , the function U satisfies the so–called Inada's conditions: it is strictly increasing, strictly concave, continuously differentiable and $U'(0) = \lim_{x\to 0+} U'(x) = +\infty$, $U'(+\infty) = \lim_{x\to+\infty} U'(x) = 0$.

After previous results by Markowitz in the context of a single period model (see [26]), the continuous time version was proved by Merton ([28, 29]) in the case where asset prices are diffusion processes with constant drift and volatility coefficients: many extensions were subsequently given in terms of various incomplete markets (and with constraints on the strategies) by several authors. See for instance [6, 20, 21, 22].

Section 2 of the present review paper gives an outline of the Merton's original method (without a complete proof) and Section 3 gives (almost as an exercise) an alternative proof based on stochastic integral representation of martingales in a Brownian filtration.

Section 4 introduces infinite dimensional models for financial markets and a theory of stochastic integration explicitely developed for the investigation of these models, while Section 5 exposes some extensions of the Mutual Fund theorem.

2. An outline of the classical proof

In this Section, we give an outline of the Merton's classical proof (based on stochastic control methods), closely following the presentation given by Bjork (see [2, chapter 19]).

The level of this section is heuristic: besides the original papers [28] and [29], the interested reader can find an accurate presentation of Merton's results (together with a concise introduction to the Stochastic Optimal Control) in the quoted book by Biork.

According to the model of Samuelson–Merton–Black–Scholes, the risky assets are supposed to satisfy the equation

$$\mathrm{d}S_t^i \;=\; S_t^i\Big(\mu^i\,\mathrm{d}t \,+\, \sum_{j=1}^{n}\sigma_{ij}\,\mathrm{d}W_t^j\Big) \tag{2.1}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ is a vector of $\mathbb{R}^n$, $\mathbf{W} = (W^1, \ldots, W^n)$ is an n–dimensional Wiener process and $\boldsymbol{\sigma} \;=\; \big[\sigma_{i,j}\big]_{i,j=1,\ldots,n}$ is a $n \times n$ invertible matrix: under these assumptions, the model is *arbitrage free* and *complete.*

By using (as in the previous Section) the *relative portfolio* $\mathbf{u}_t = (u_t^1, \ldots, u_t^n)$ as a *control*, the equation of the corresponding porfolio value is

$$\mathrm{d}X_t^{\mathbf{u}} \;=\; X^{\mathbf{u}}\left(\mathbf{u}_t.\boldsymbol{\mu}\,\mathrm{d}t \,+\, \boldsymbol{\sigma}^*\mathbf{u}_t.\mathrm{d}\mathbf{W}_t\right) \tag{2.2}$$

Therefore, $X_t^{\mathbf{u}}$ is a *diffusion process* with *infinitesimal generator*

$$\mathcal{A}_t^{\mathbf{u}} \quad=\quad x\,\mathbf{u}.\boldsymbol{\mu}\,\frac{\partial}{\partial x} \,+\, \frac{x^2}{2}\,\|\boldsymbol{\sigma}^*\mathbf{u}\|^2\,\frac{\partial^2}{\partial x^2}$$

As it is usual in stochastic optimal control, one considers the *optimal value function*

$$V(t,x) \;=\; \sup_{\mathbf{u}\in\mathcal{U}} \mathbb{E}\Big[U\Big(X_T^{t,x,\mathbf{u}}\Big)\Big]$$

where $\mathcal{U}$ is the class of admissible controls (in this case, *all* controls) and $X^{t,x,\mathbf{u}}$ is the process which starts from x at time t and follows the dynamics given by (2.2).

Under suitable assumptions (obviously satisfied in this simple model with constant coefficients) the function V is the solution of the Hamilton–Jacobi–Bellman equation

$$\begin{cases} \frac{\partial V}{\partial t} \,+\, \sup_{\mathbf{u}\in\mathbb{R}^d}\Big[\mathcal{A}^{\mathbf{u}}V(t,x)\Big] \;=\; 0 \\ V(T,x) \;=\; U(x) \end{cases}$$

Handling the HJB equation in practice, is given in two steps:

- given (t,x) and the function V, find $\hat{\mathbf{u}}(t,x,V)$ solution of

$$\mathcal{A}^{\hat{\mathbf{u}}}V(t,x) \;=\; \max_{\mathbf{u}\in\mathbb{R}^n}\left[\mathcal{A}^{\mathbf{u}}V(t,x)\right]$$

- solve the equation

$$\begin{cases} \frac{\partial V}{\partial t} \,+\, \mathcal{A}^{\hat{\mathbf{u}}(t,x,V)}V(t,x) \;=\; 0 \\ V(T,x) \;=\; U(x) \end{cases}$$

The solution of $\arg\max_{\mathbf{u}\in\mathbb{R}^n}\Big(x\,\boldsymbol{\mu}.\mathbf{u}\,V_x + \frac{x^2}{2}\|\boldsymbol{\sigma}^*\mathbf{u}\|^2\,V_{xx}\Big)$ is given by $\hat{\mathbf{u}} = \frac{-V_x}{x V_{xx}}\,\big(\boldsymbol{\sigma}\boldsymbol{\sigma}^*\big)^{-1}\boldsymbol{\mu}$.

Before summarizing these results in a complete statement, denote by $a = \sum_{i=1}^n \big((\boldsymbol{\sigma}\boldsymbol{\sigma}^*)^{-1}\boldsymbol{\mu}\big)_i$ and $\mathbf{f} = \frac{(\boldsymbol{\sigma}\boldsymbol{\sigma}^*)^{-1}\boldsymbol{\mu}}{a}$. We have the following Theorem (see [2, Theorem 19.10]):

Theorem 2.1 (Mutual Fund Theorem). *The optimal portfolio is an allocation between the riskless asset and a* fund *(more precisely a* portfolio*) which consists only of risky assets and corresponds to the control* $\mathbf{f}$.

At each time t, the relative allocation of wealth between the fund and the riskless asset is given by $m^f(t) = -\frac{a\,V_x(t,X_t)}{X_t\,V_{xx}(t,X_t)}$ *and* $m^0(t) = 1 - m^f(t)$.

In this simple situation with constant deterministic coefficients (the model investigated by Merton) the solution of the H.J.B. equation is classical, but in more general situations the solution has to be understood in the viscosity sense. For a comprehensive presentation of recent advanced results in this direction the reader can be addressed to the two interesting courses at *"Scuola Normale Superiore"* given by N. Touzi and M. Soner (see [33] and [31]).

3. A proof based on stochastic analysis

From now on, we prefer to use the process $\mathbf{H}_t$ (as defined in Section 1) for the representation of the strategy, rather than the relative portfolio.

The starting point of this approach is that, if we indicate by $\hat{X}(x)$ the optimal solution of the utility maximization problem, then $U'\big(\hat{X}(x)\big)$ is proportional to the density of the *equivalent martingale probability* $\big(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\big)$.

The *intuition* for this statement can be given as follows: if $\mathbf{K}_s$ is another n–dimensional predictable process and we consider the strategy $\big(\mathbf{H}_s + t\,\mathbf{K}_s\big)$, we have

$$\mathbb{E}\Big[U\Big(\hat{X}(x) + t\int_0^T \mathbf{K}_s \mathrm{d}\mathbf{S}_s\Big)\Big] \le \mathbb{E}\Big[U\Big(\hat{X}(x)\Big)\Big]$$

and hence the derivative with respect to t, for $t = 0$, has to be 0. More precisely

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathbb{E}\Big[U\Big(\hat{X}(x) + t\int_0^T \mathbf{K}_s \mathrm{d}\mathbf{S}_s\Big)\Big] = \mathbb{E}\Big[U'\Big(\hat{X}(x)\Big)\,.\int_0^T \mathbf{K}_s \mathrm{d}\mathbf{S}_s\Big]$$

whatever is the strategy $\mathbf{K}$ (provided that suitable integrability conditions are satisfied): necessarily $U'\big(\hat{X}(x)\big)$ (which is a positive r.v.) is proportional to $\big(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\big)$.

Obviously this intuition needs a rigorous proof: the most general formulation (in the framework of incomplete markets) is given in [24].

Let us write the equation (2.1) in a vector form: given $\mathbf{x} \in \mathbb{R}^n$, we indicate by $D[\mathbf{x}]$ the diagonal matrix $D[\mathbf{x}] = \mathrm{diag}[x^1, \ldots, x^n]$.

The equation (2.1) can be rewritten as

$$d\mathbf{S}_t = D\big[\mathbf{S}_t\big]\big(\boldsymbol{\mu}\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\mathbf{W}_t\big) = D\big[\mathbf{S}_t\big]\boldsymbol{\sigma}\mathrm{d}\big(\mathbf{W}_t + \boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\, t\big) = D\big[\mathbf{S}_t\big]\boldsymbol{\sigma}\mathrm{d}\mathbf{W}^*_t \tag{3.1}$$

The process $\mathbf{W}^*_t = \mathbf{W}_t + \boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\, t$ is a n–dimensional Wiener process under the probability $\mathbf{Q}$ given by the formula

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \exp\Big(-\int_0^T \boldsymbol{\sigma}^{-1}\boldsymbol{\mu}.\mathrm{d}\mathbf{W}_s - \frac{1}{2}\int_0^T \|\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\|^2\mathrm{d}s\Big)$$

Consider the scalar process $Z_t = \frac{\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}}{\|\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\|}.\mathbf{W}^*_t$: Z is a one–dimensional $\mathbf{Q}$–Wiener process and $\hat{X}(x) = \big(U'\big)^{-1}\big(y\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\big)$ is measurable with respect to the filtration generated by $\big(Z_t\big)_{0\le t\le T}$. Therefore we have the equality $\hat{X}(x) = x + \int_0^T \gamma_s \mathrm{d}Z_s$, where γ_s is a suitable scalar predictable process.

The equation (3.1) can be rewritten in the form

$$\mathrm{d}\mathbf{W}^*_t = \boldsymbol{\sigma}^{-1}D\big[\frac{1}{\mathbf{S}_t}\big].\mathrm{d}\mathbf{S}_t$$

We have therefore

$$\begin{aligned}\hat{X}(x) &= x + \int_0^T \frac{\gamma_s}{\|\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\|}\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}.\mathrm{d}\mathbf{S}_s = x + \int_0^T \frac{\gamma_s}{\|\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\|}\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\boldsymbol{\sigma}^{-1}D\big[\frac{1}{\mathbf{S}_s}\big].\mathrm{d}\mathbf{S}_s \\ &= x + \int_0^T \frac{\gamma_s}{\|\boldsymbol{\sigma}^{-1}\boldsymbol{\mu}\|}\big(\boldsymbol{\sigma}\boldsymbol{\sigma}^*\big)^{-1}\boldsymbol{\mu}\, D\big[\frac{1}{\mathbf{S}_s}\big].\mathrm{d}\mathbf{S}_s\end{aligned}$$

The result of Theorem 2.1 can be rewritten in this form: for every (ω, t), the optimal *relative portfolio* $\mathbf{u}_t(\omega)$ is proportional to the vector $\big(\boldsymbol{\sigma}\boldsymbol{\sigma}^*\big)^{-1}\boldsymbol{\mu}$ and this is equivalent to say that the *optimal strategy* $\mathbf{H}_t(\omega)$ is proportional to $\big(\boldsymbol{\sigma}\boldsymbol{\sigma}^*\big)^{-1}\boldsymbol{\mu}\, D\big[\frac{1}{\mathbf{S}_t(\omega)}\big]$. So we have obtained the *mutual fund theorem.*

In order to extend this method of proof to more general situations, it is worth pointing out the essential steps:

- the value of the optimal portfolio $\hat{X}(x)$ exists and is equal to $\big(U'\big)^{-1}\big(y\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\big)$ with a suitable positive constant y;
- the density of the equivalent martingale probability is measurable with respect to a *smaller filtration* $\big(\mathcal{G}_t\big) \subseteq \big(\mathcal{F}_t\big)$ and on this filtration there is a *stochastic integral representation property* with respect to a (k–dimensional) $\mathbf{P}$–martingale $(N_t)_{0\le t\le T}$;
- the martingale (N_t) can be written as the value of a portfolio (and identifies the *mutual fund*).

Concerning the first statement, we have a general result given by Kramkov–Schachermayer (see [24] Thm 2.0 for details): let us first define the set of the so-called *equivalent martingale measures.*

Definition 3.1. We indicate by $\mathcal{M}$ the set of all equivalent probabilities $\mathbf{Q}$ with the property that, for every strategy $\mathbf{H}$, if the process $Y_t = \int_0^t \mathbf{H}_s\mathrm{d}\mathbf{S}_s$ is uniformly bounded from below, then it is a $\mathbf{Q}$–supermartingale.

It is usually assumed that the set $\mathcal{M}$ is non-empty: this is in some sense equivalent to an *Absence of Arbitrage* condition (see [12] and [13] for a more precise formulation).

The result stated in [24] is the following: if the market is *complete* (more precisely, if the set $\mathcal{M}$ is a singleton) then $\hat{X}(x)$ exists and is equal to $(U')^{-1}\left(y\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right)$ (with a suitable positive constant y), for every positive x if the utility function U satisfies an additional property (reasonable asymptotic elasticity), and given a general utility function U if x is not too big.

4. Infinite-dimensional financial models

There are two situations, in stochastic models for Finance, where infinite–dimensional models are used: Large Financial Markets and Bond Markets.

Large Financial Markets were modeled in [4] as markets containing an infinite, countable, set of traded assets, represented by a sequence of semimartingales $\left(S^n_t\right)_{0\le t\le T}$, $n = 0, 1, \ldots$ on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P}\right)$.

In the Bond Market models, it is conventional to assume that at every time $t \ge 0$ there exists a bond $P(t,T)$ that matures at time T for $t \le T \le T^*$: we have in this case a *continuum* of stochastic processes $\left(P(t,T)\right)_{0\le t\le T\le T^*}$.

From the point of view of *infinite dimensional stochastic integration*, much attention has been devoted to Bond Market models: see for instance [3, 5, 15].

The usual approach is to model $P(t,.)$ as a stochastic process with values in a suitable (Hilbert) space $\mathcal{H}$ of continuous functions defined on $[0,T^*]$: for instance, in the papers [5] or [15], $\mathcal{H}$ is an appropriate weighted Sobolev space. The natural space where the integrands should take values is the dual space $\mathcal{H}'$, and the quoted papers contain an adaptation of results of infinite dimensional stochastic integration.

A different approach was investigated by Bjork et al. (see [3]): they consider the Bond price process as a stochastic process with values in the space of continuous functions on $[0,T^*]$, and develop a theory of stochastic integration where the integrand ϕ_t takes values in the space of signed Radon measures on $[0,T^*]$.

A different method was introduced by M. De Donno and the author in the papers [10] for the case of a sequence of semimartingales and [11] for the case of Bond Markets: we shall expose this approach more in details.

Let I a set and consider an indexed family $\left(S^x_t\right)_{x\in I}$ of semimartingales defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P}\right)$: in our applications, I will be $\mathbb{N}$ or $[0,T^*]$ (and in the second case we impose that the application $x \to S^x$ is continuous with respect to the topology of semimartingales introduced by Émery in [17]).

We consider $\mathbf{S} = (S^x)_{x\in I}$ as a stochastic process with values in the product space $\mathbb{R}^I$: when the latter is endowed with the product topology, its dual space is formed by the finite linear combinations of Dirac's deltas (δ_x).

We call *simple integrand* a process $\mathbf{H}$ of the form $\mathbf{H}(\omega,t)=\sum_{i\le n} H^i(\omega,t)\,\delta_{x_i}$, where $x_1,\dots,x_n\in I$ and every H^i is a scalar bounded predictable process: given a simple integrand $\mathbf{H}$, it is natural to define the *stochastic integral*

$$\int_{]0,t]} \mathbf{H}_s\,\mathrm{d}\mathbf{S}_s \;=\; \int_{]0,t]} \sum_{i\le n} H^i_s\,\mathrm{d}S^i_s \tag{4.1}$$

Note that a simple integrand is the mathematical counterpart of a real world portfolio, which is based on a finite number of assets.

In order to obtain a larger class of integrands, it is convenient to introduce processes with values in the set of non-continuous (unbounded) linear functionals on $\mathbb{R}^I$. Denoting by $\mathcal{U}$ the set of these unbounded functionals, we give the following definition:

Definition 4.1. Let $\mathbf{H}$ be a $\mathcal{U}$-valued process. We say that $\mathbf{H}$ is *integrable* with respect to $\mathbf{S}$ if there exists a sequence $(\mathbf{H}^n)$ of simple integrands such that

(i) $\mathbf{H}^n$ converges to $\mathbf{H}$ a.s.;
(ii) $\left(\int \mathbf{H}^n_s\mathrm{d}\mathbf{S}_s\right)$ converges to a semimartingale Y for the semimartingale topology.

We call $\mathbf{H}$ a *generalized integral* and define $\int \mathbf{H}\,\mathrm{d}\mathbf{S}=Y$.

The above definition needs some explanations: the statement (i) means that, for a.e. (ω,t), if $x\in \mathrm{Dom}\,\mathbf{H}(\omega,t)$, then $\mathbf{H}^n(\omega,t)(x)$ converges to $\mathbf{H}(\omega,t)(x)$. Almost surely means *outside of a set negligible for every semimartingale* S^x: a more precise and formal definition can be found in [10] and [11].

It is clear that Definition 4.1 makes sense only provided that the limit semimartingale Y does not depend on the approximating sequence: this was proved in [10] (Proposition 5.1) for the case of a sequence of semimartingales and [11] (Proposition 2.3) for the case of Bond Market models.

We wish also to point out that the Definition 4.1 of integrable process is suggested by the notion of integrable function with respect to a vector-valued measure (see [12], section IV.10.7).

In order to compare this approach of infinite-dimensional stochastic integration with the previously cited approaches, let un point out that in the finite-dimensional case a fundamental result is the following:

Proposition 4.2. *Let f be a positive function: f satisfies an inequality of the form $f\le x+\int_0^T \mathbf{H}_s\mathrm{d}\mathbf{S}_s$ (with a suitable admissible strategy $\mathbf{H}$ and a positive constant x) if and only if, for every $\mathbf{Q}\in\mathcal{M}$, one has $\mathbb{E}^{\mathbf{Q}}\big[f\big]\le x$.*

The result of Proposition 4.2 was proved by El Karoui and Quenez (see [16]) in the case of diffusion processes, and by Delbaen-Schachermayer in the general semimartingale framework (see [12], and also [13] for a comprehensive presentation). It is worth pointing out that this result is strictly linked to the so called *optional decomposition* (proved, in the general semimartingale case, by D. Kramkov [23]): in fact the optional decomposition is a more general result (the paper [30] by H.

Pham gives an infinite-dimensional version of this decomposition, in the framework of jump-diffusion processes).

Proposition 4.2 is an essential step in the convex duality approach to the *utility maximization problem*, along the lines of the general papers by Kramkov and Schachermayer ([24] and [25]). The very technical proof is based on two properties of the (finite-dimensional) stochastic integrals:

(a) the so-called *Memin's theorem*, which states that limit of stochastic integrals (for the semimartingale topology) is still a stochastic integral;
(b) the *Ansel-Stricker's lemma*, which states that, if $\mathbf{M}$ is a local martingale, $\mathbf{H}$ is $\mathbf{M}$-integrable and the stochastic process $\int_0^t \mathbf{H}_s \mathrm{d}\mathbf{M}_s$ is uniformly bounded from below, then it is a *supermartingale*.

The extension of (a) is not satisfied by the approaches given e.g. by Carmona-Tehranchi or Ekeland-Taflin, while is satisfied with Definition 4.1. More precisely we have the following result (see [10] and [11]):

Theorem 4.3. *Let* $\mathbf{H}^n$ *be a sequence of generalized integrands such that* $\left(\int \mathbf{H}^n \mathrm{d}\mathbf{S}\right)$ *is a Cauchy sequence in the space of semimartingales: then there exists a generalized integrand* $\mathbf{H}$ *such that* $\lim_{n\to\infty} \int \mathbf{H}^n \mathrm{d}\mathbf{S} = \int \mathbf{H} \mathrm{d}\mathbf{S}$.

Unfortunately, the Ansel-Stricker's lemma is false for generalized integrands (see [10] and [11] for counterexamples). Therefore the definition of *admissible strategy* has to be modified in the following way:

Definition 4.4. A generalized integrand $\mathbf{H}$ is called an admissible strategy if there exist a constant x and a sequence of approximating elementary integrands $\mathbf{H}^n$ such that:

(i) $\int_0^t \mathbf{H}^n_s \mathrm{d}\mathbf{S}_s \geq x$ a.s. for every t;
(ii) the sequence $\int \mathbf{H}^n \mathrm{d}\mathbf{S}$ converges to $\int \mathbf{H} \mathrm{d}\mathbf{S}$ for the semimartingale topology.

With this definition of admissible strategy, the results of Proposition 4.2 and the convex duality approach of [24] and [25] can be extended to infinite-dimensional models: see [9] for the case of Large Financial Markets and [11] for Bond Market models.

It is worth pointing out that there are different papers which investigate, by different methods, the problem of *utility maximization* within a Bond Market model: these are, for instance, the papers by Ekeland-Taflin (see [15]) or Ringer-Tehranchi ([32]). The latter paper, in particular, obtains a *mutual fund theorem*.

5. Generalizations of the Mutual Fund theorem

Let us first insist more on the *No Arbitrage* conditions for an Infinite Dimensional model. When we have an infinite family of semimartingales $\left(S^x\right)_{x\in I}$, we indicate by $\mathcal{M}$ the set of all equivalent probabilities $\mathbf{Q}$ such that, for every finite subset

$(x_1, \ldots, x_n) \subset I$, the property described in Definition 3.1 is satisfied by the n-dimensional semimartingale $\left(S^{x_1}, \ldots, S^{x_n}\right)$: we suppose that the set $\mathcal{M}$ is non-empty and we say that the market is *complete* when $\mathcal{M}$ is a singleton.

The integral defined in Section 4 (Definition 4.1), which satisfies a sort of *Memin's theorem* (Thm. 4.3) is a good mathematical tool in order to face the *utility maximization problem* in an infinite dimensional market; and when the model satisfies the properties listed at the end of Section 3, it is natural to expect that a *mutual fund theorem* can be obtained.

For instance, the paper [9] contains such a theorem for the case of Large Financial Markets, and [8] a similar result for the case of Bond Markets. Rather than to enumerate such results, we prefer to develop an example in the case of Large Financial Markets.

Let us first mention that these models were introduced by Kabanov and Kramkov (see [18] and [19]) in order to study the existence (or non existence) of *Asymptotic Arbitrage* possibilities: to this aim, they model a Large Financial Market as a sequence of finite-dimensional financial models.

But problems such as *completeness* or *pricing of derivatives* are hard to study in this framework: to this extent, Bjork and Näslund (see [4]) choose to model a Large Financial Market as a sequence of semimartingales defined on a fixed filtered probability space and investigate the consequences of *diversification* of risk sources.

Let us examine in more details a *Factor Model* as introduced in [4]. We assume that every asset price depends on a systematic source of randomness which affects all the assets and on a idiosyncratic source of randomness which is typical for that asset. In particular, we assume that the price processes evolve according to the following dynamics:

$$dS_t^i = S_{t-}^i \left(\alpha_i dt + \beta_i d\hat{N}_t + \sigma_i dW_t^i\right)$$

where $(W^i)_{i\geq 1}$ is a sequence of independent Wiener processes and $\hat{N}_t = N_t - \lambda t$ is a compensated Poisson process with intensity λ (independent of W^i for all i). The Poisson process models some shocks which may occur in the market and may affect all the assets. As in [4], the coefficients $\alpha_i, \beta_i, \sigma_i$ are constants: in particular we assume that $\beta_i, \sigma_i \geq \epsilon > 0$ for all i and that there exists M such that $\sup_i(|\alpha_i|, \beta_i, \sigma_i) \leq M$.

Björk and Näslund studied the questions of No Arbitrage and completeness and showed that an asymptotic *well diversified* portfolio can be defined (as limit of a sequence of portfolios based on the first n assets), in order to complete the market. The intuitive notion of well diversified portfolio can be translated in a more formal way into the definition of *generalized integrand* given in Section 4: a thorough investigation of completeness (via the integral defined in the previous Section) was given by M. De Donno in [7]. Here, we want to analyze the problem of utility maximization in order to obtain a *mutual fund theorem.*

We take as filtration $(\mathcal{F}_t)_{t\leq T}$ the (completed) filtration generated by the price processes, hence by $\{(W^i)_{i\geq 1}, N\}$. It is well-known that every local martingale L

has necessarily the form

$$L_t = L_0 + \int_0^t K_s d\hat{N}_s + \sum_{i\geq 1} \int_0^t H_s^i dW_s^i, \tag{5.1}$$

where $K, (H^i)_{i\geq 1}$ are predictable processes and

$$\int_0^T |K_s| ds + \sum_{i\geq 1} \int_0^T (H_s^i)^2 ds < \infty \qquad \text{a.s.} \tag{5.2}$$

Let $\mathbf{Q}$ be a probability measure equivalent to $\mathbf{P}$. Then, its density has the form $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P} = \mathcal{E}(L_T)$ (we recall that $\mathcal{E}$ denotes the stochastic exponential), where L has the form (5.1), with $L_0 = 0$. Furthermore, we have that $K_s > -1$ in order to ensure that $\mathcal{E}(L_1) > 0$ and L is such that $\mathcal{E}(L_t)$ is a uniformly integrable martingale.

By Girsanov's theorem, it follows that the process $\tilde{W}_t^i = W_t^i - \int_0^t H_s^i \,\mathrm{d}s$ is a $\mathbf{Q}$-Wiener process, while the process $\tilde{N}_t = \hat{N}_t - \int_0^t K_s \,\mathrm{d}s = N_t - \int_0^t (1 + K_s)\,\mathrm{d}s$ is a $\mathbf{Q}$-martingale (namely $\int_0^t (1 + K_s)\,\mathrm{d}s$ is the $\mathbf{Q}$-compensator of the point process N).

Since every $(S^i)_{i\geq 1}$ is locally bounded, we have that $\mathbf{Q} \in \mathcal{M}$ if and only if S^i is a $\mathbf{Q}$-local martingale and this occurs if and only if

$$H_t^i = -\frac{\alpha_i + \beta_i K_t}{\sigma_i}$$

for all $i \geq 1$. Then, by condition (5.2), it must be $\int_0^T \sum_i (\alpha_i + \beta_i K_t)^2 \sigma_i^{-2} dt < \infty$: it is easy to check that this implies that the sequence (α_i/β_i) converges to some real number h_0. This implies that $K_t = \frac{-h_0}{\lambda} = k$, $H_t^i = \frac{-(\alpha_i+\beta_i h_0)}{\sigma_i} = h^i$ and that there exists a unique equivalent martingale measure $\mathbf{Q}$, provided that $h_0 < \lambda$ (the uniform integrability of the martingale $\mathcal{E}(L_t)$ is a consequence of Novikov condition).

Conversely, on the n-dimensional market, there are infinitely many equivalent martingale measures. In particular, the point process N may have any intensity, and, possibly, even a stochastic compensator. We can see immediately the difference among every finite (n-dimensional) market and the large (infinite dimensional) market:

- every n-dimensional market is incomplete, while the large market is complete;
- in every n-dimensional market the utility maximization problem is difficult to solve and there is not a mutual fund theorem, while in the large market the problem becomes easy and we have a mutual fund theorem.

Let us see more in detail the proof of the last sentence. As in Section 3, the value $\hat{X}(x)$ of the optimal portfolio can be written in the form $\big(U'\big)^{-1}\big(y\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\big)$ with a suitable positive constant y. Note that

$$\begin{aligned}\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \mathcal{E}(L_T) &= \mathcal{E}\left(\sum_{j\geq 1} h_j W_T^j - h_0 \hat{N}_T\right) \\ &= \exp\left(T\sum_{i\geq 0} h_i^2\right)\mathcal{E}\left(\sum_{j\geq 1} h_j \tilde{W}_T^j - h_0 \tilde{N}_T\right)\end{aligned}$$

Denote by $\tilde{W}_h$ the process $\sum_{j\geq 1} h_j \tilde{W}^j$. This is a brownian motion with respect to the probability $\mathbf{Q}$ as well as the process $\tilde{N}$ is a $\mathbf{Q}$-compensated Poisson process (with compensator $\lambda(1 - h_0/\lambda)\, t = (\lambda - h_0)\, t$). Furthermore, both $\tilde{W}_h$ and $\tilde{N}$ coincide with the values of two self-financing portfolios: more precisely, there exists a pair of generalized strategies $\mathbf{H}^1$ and $\mathbf{H}^2$ such that

$$\tilde{W}_h = \int \mathbf{H}^1 \mathrm{d}\mathbf{S}\,, \qquad \tilde{N} = \int \mathbf{H}^2 \mathrm{d}\mathbf{S}\,. \tag{5.3}$$

This is a consequence of market completeness, for more details one can consult [7].

Observe that $\tilde{W}_h$ and $\tilde{N}$ can be interpreted as mutual funds, each composed of a small part of each asset. In particular $\tilde{W}_h$ does not depend on the systematic risk and contain a small part of all the idiosyncratic risks, while $\tilde{N}$ is based only on the systematic risk.

$\hat{X}(x)$ is measurable with respect to the filtration generated by $(\tilde{W}_h, \tilde{N})$, hence it admits a representation as:

$$\hat{X}(x) = x + \int_0^T \phi_s(x) d(\tilde{W}_h)_s + \int_0^T \psi_s(x) d\tilde{N}_s$$

This, combined with (5.3), allows us to find the optimal strategy $\hat{\mathbf{H}}(x) = \phi(x)\mathbf{H}^1 + \psi(x)\mathbf{H}^2$. Note that $\mathbf{H}^1$ and $\mathbf{H}^2$ depend only on the density of the equivalent martingale measure, while $\phi(x)$ and $\psi(x)$ are the sole processes affected by the choice of the utility function. So, we can claim a mutual fund theorem:

Theorem 5.1. *For any utility function U, the optimal portfolio consists of an allocation between the risk free asset, the mutual fund $\tilde{W}_h$ and the mutual fund $\tilde{N}$.*

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