Lezione 12 - Cambio di numerario nei modelli per i tassi d'interesse...

$(\Omega, \mathcal{F}, P)$ sotto P^* $\forall T$ $\left.\frac{B(t,T)}{B_t}\right|_{0 \le t \le T}$ è una martingala

Nuovo numerario $B(t,T)$ $P^T \leftarrow$ T-forward measure

$$\left.\frac{dP^T}{dP^*}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{B_t \cdot B(0,T)} \quad \leftarrow \text{è definito su } \mathcal{F}_T \text{ "fino al tempo } T\text{"}$$

$$\left.\frac{dP^T}{dP^S}\right|_{\mathcal{F}_t} = \frac{B(t,T)}{B(t,S)} \cdot \frac{B(0,S)}{B(0,T)} \qquad t \le \min(T,S)$$

attivo finanziario X al tempo S

prezzo equo in t $\quad B_t E^*\left[\frac{X}{B_S} \middle| \mathcal{F}_t\right] = B(t,S)\, E^S\left[X \middle| \mathcal{F}_t\right] =$

$$= B(t,T)\, E^T\left[\frac{X}{B(S,T)} \middle| \mathcal{F}_t\right]$$

dinamica dei bond sotto P^*

$$dB(t,T) = B(t,T)\left(r(t)\,dt + S(t,T)\,dW_t\right)$$

$$d\left(\frac{B(t,T_2)}{B(t,T_1)}\right) = \frac{B(t,T_2)}{B(t,T_1)}\Big[(\dots)dt + \underbrace{\left(S(t,T_2) - S(t,T_1)\right)}_{\sigma(t,T_1,T_2)} dW_t\Big]$$

se W_t è a più dimensioni

$$\sigma(t,T_1,T_2) \circ dW_t = \|\sigma(t,T_1,T_2)\| \cdot d\tilde{W}_t$$

Wiener mono-dim.

se è deterministico modello log-normale

Opzione Call al tempo S su un bond di scadenza T

$$C_t = B_t E^*\left[\frac{(B(S,T)-k)^+}{B_S} \middle| \mathcal{F}_t\right] = B(t,S)\, E^S\left[(B(S,T)-k)^+ \middle| \mathcal{F}_t\right] =$$

$$A = \{B(S,T) > k\} = \left\{\frac{B(S,T)}{B(S,S)} > k\right\}$$

$$= B(t,S)\, E^S\left[B(S,T) I_A \middle| \mathcal{F}_t\right] - k B(t,S)\, E^S\left[I_A \middle| \mathcal{F}_t\right] =$$

$$= B(t,T)\, E^T\left[I_A \middle| \mathcal{F}_t\right] - k B(t,S)\, E^S\left[I_A \middle| \mathcal{F}_t\right]$$

$$d\left(\frac{B(t,S)}{B(t,T)}\right) = (\quad)\left[\dots dt + \sigma(t,S,T)\,dW_t\right]$$

se è deterministico

$$C_t = B(t,T)\,\Phi(d_1) - k B(t,S)\,\Phi(d_2)$$

$$d_{1,2} = \frac{\log\left(\frac{B(t,T)}{B(t,S)\,k}\right) \pm \frac{1}{2}\Sigma^2_{S,T}(t)}{\sqrt{\Sigma^2_{S,T}(t)}}$$

$$\Sigma^2_{S,T}(t) = \int_t^S \|\sigma(s,S,T)\|^2\,ds$$

portafoglio di replicazione basato su $B(t,S)$ e $B(t,T)$

si può replicare	si può replicare
$B_t \leftarrow$ money acc.	Bond e 3 e più non
$B(t,\tilde{T})$ $\tilde{T} > S$ qualsiasi	

Modelli basati su $r(t)$ dotati di A.T.S.

$$dr(t) = \left(\alpha(t) r(t) + \beta(t)\right)dt + \sqrt{\gamma(t) r(t) + \delta(t)}\,dW_t$$

$$B(t,T) = \exp\left(a(t,T) - b(t,T)\, r(t)\right)$$

$$d\left(\frac{B(t,S)}{B(t,T)}\right) = \left(\frac{B(t,S)}{B(t,T)}\right)\Big[\dots dt + \underbrace{\left(b(t,T) - b(t,S)\right)\sqrt{\gamma(t) r(t) + \delta(t)}}_{\sigma(t,S,T)}\, dW_t\Big]$$

modello log-normale

modello C.I.R. non è log-normale

~~portafoglio~~ repl. $B(t,S)$ e $B(t,T)$

$\mathcal{F}^{\dots} = \mathcal{F}^W$

Modelli tipo H.J.M.

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t$$

$$S(t,T) = -\int_t^T \sigma(t,s)\,ds$$

$$\sigma(t,S,T) = -\int_S^T \sigma(t,s)\,ds$$

se è deter.

$$\Sigma^2_{S,T}(t) = \int_t^S \left\|\int_S^T \sigma(s,u)\,du\right\|^2 ds$$

$$\sigma(t,T) = \left(\sigma_1, \sigma_2 e^{-\lambda(T-t)}\right)$$

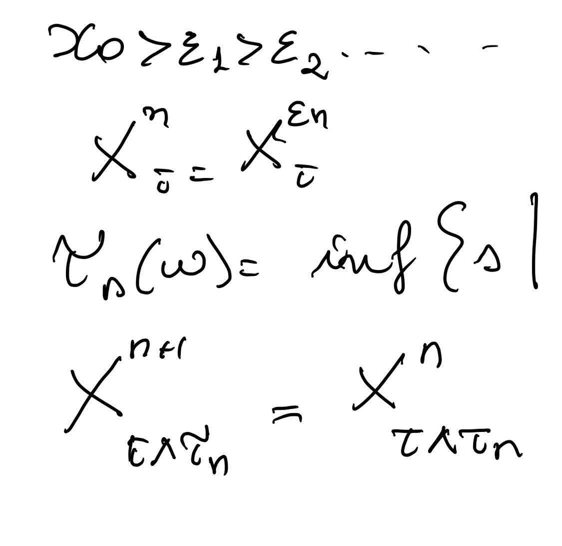
Dimostrazione rigorosa del modello C.I.R.

$$dr(t) = a(b - r(t))\,dt + \sigma\sqrt{r(t)}\,dW_t$$

$$r(0) = r^*(0)$$

$$\begin{cases} dX_t = \mu(t, X_t)\,dt + \sigma\sqrt{X_t}\,dW_t \\ X_0 = x_0 > 0 \end{cases} \qquad X(\omega,t) > 0$$

$0 < \varepsilon < x_0$



$$g_\varepsilon(x) = \begin{cases} \sqrt{x} & x > \varepsilon \\ C^1 & \text{tra} \\ 0 & x < 0 \end{cases}$$

$$\begin{cases} dX^\varepsilon_t = \mu(t, X^\varepsilon_t)\,dt + g_\varepsilon(X^\varepsilon_t)\,dW_t \\ X^\varepsilon_0 = 0 \end{cases}$$

$x_0 > \varepsilon_1 > \varepsilon_2 \dots \qquad \varepsilon_n \downarrow 0$

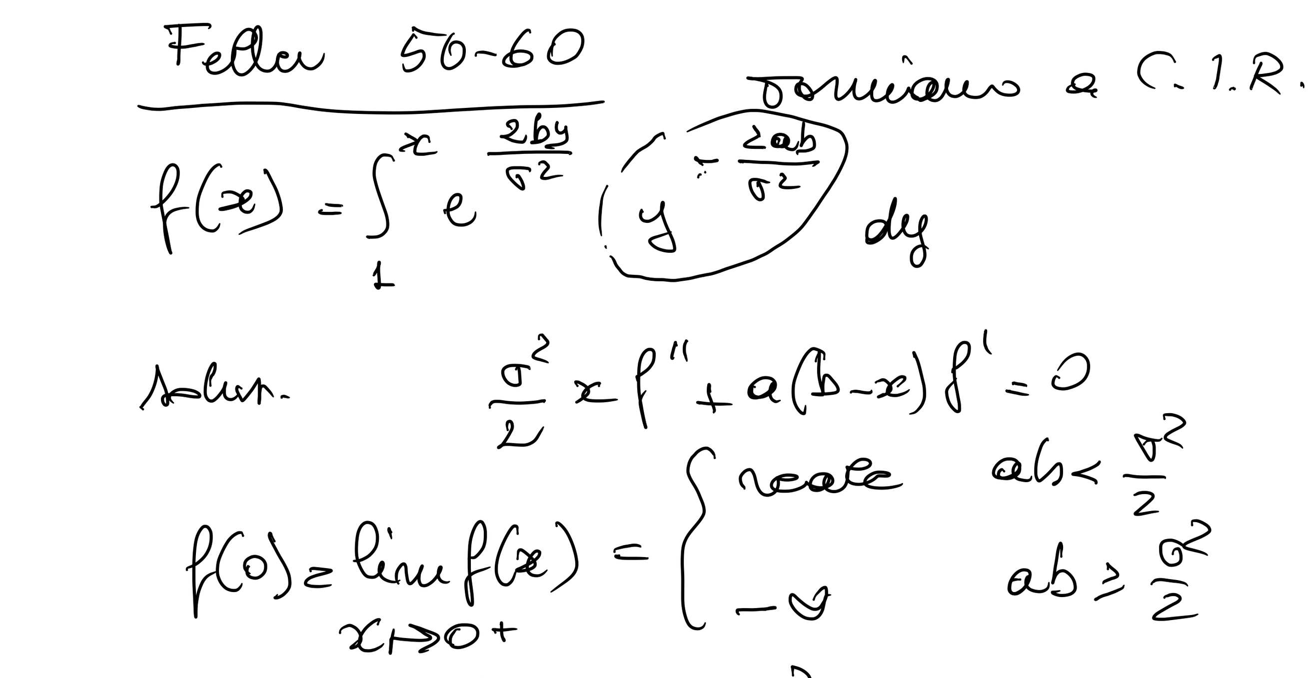
$X^n_t = X^{\varepsilon_n}_t$

$\tau_n(\omega) = \inf\{s \mid X^{n+1}(\omega,s) < \varepsilon_n\} \qquad \inf \emptyset = +\infty$

$X^{n+1}_{t \wedge \tau_n} = X^n_{t \wedge \tau_n} \qquad \tau_n \le \tau_{n+1} \dots \qquad \tau = \lim_n \tau_n$

soluzione definita su

$$[0,\tau[\, = \{(\omega,t) \mid \omega \in \Omega,\ 0 \le t < \tau(\omega)\}$$



$\tau(\omega) = +\infty$ — soluzione globale

$\tau(\omega) < +\infty$ " parziale

Feller 50-60

tornando a C.I.R.

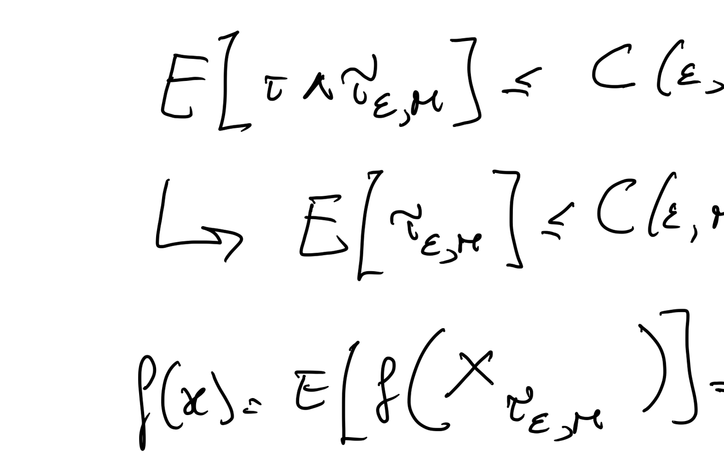
$$f(x) = \int_1^x e^{\frac{2by}{\sigma^2}}\, y^{-\frac{2ab}{\sigma^2}}\,dy$$

solut. $\quad \frac{\sigma^2}{2} x f'' + a(b-x) f' = 0$

$$f(0) = \lim_{x \to 0^+} f(x) = \begin{cases} \text{reale} & ab < \frac{\sigma^2}{2} \\ -\infty & ab \ge \frac{\sigma^2}{2} \end{cases}$$

$\tau_\varepsilon(\omega) = \inf\{s : X_s \le \varepsilon\}$

$\tau_M(\omega) = \inf\{s : X_s \ge M\} \qquad \tau_{\varepsilon,M} = \tau_\varepsilon \wedge \tau_M$



$$f(X_{t \wedge \tau_{\varepsilon,M}}) = f(x) + \int_0^{t \wedge \tau_{\varepsilon,M}} \left(f'(X_s)\,\sigma\sqrt{X_s}\right) dW_s$$

$$E\left[f(\ )^2\right] = f(x)^2 + E\left[\int_0^{t \wedge \tau_{\varepsilon,M}} (\text{—})^2\,ds\right]$$

$$E\left[t \wedge \tau_{\varepsilon,M}\right] \le C(\varepsilon,M)$$

$\hookrightarrow E\left[\tau_{\varepsilon,M}\right] \le C(\varepsilon,M) \qquad \tau_{\varepsilon,M}(\omega) < +\infty$ sempre

$$f(x) = E\left[f(X_{\tau_{\varepsilon,M}})\right] = f(\varepsilon) P\{\tau_\varepsilon < \tau_M\} + f(M) P\{\tau_\varepsilon > \tau_M\}$$

se $\tau(\omega) = +\infty$ $\forall \varepsilon$ $\Rightarrow \lim_{M \to \infty} \tau_M(\omega) = +\infty$

$\boxed{ab \ge \frac{\sigma^2}{2}} \qquad f(\varepsilon) \to -\infty$

$\lim_{\varepsilon \to 0^+} P\{\tau_\varepsilon < \tau_M\} = 0 \Rightarrow \tau = +\infty$ q.c.

$ab < \frac{\sigma^2}{2} \qquad \lim_{M \to \infty} f(M) = +\infty$

$\lim P\{\tau_\varepsilon > \tau_M\} = 0$

$\tau(\omega) < +\infty$ q.c.