Ist. Mat. I - CIA
23/5/24

Prove 1 (10/5/24)

(2) Base ortonorm. $3x-2y+z=0$

$$\begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} = u_1$$

$$\begin{pmatrix}2\\3\\0\end{pmatrix} - \frac{3}{5}\cdot\begin{pmatrix}0\\1\\2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}10\\12\\-6\end{pmatrix} = \frac{2}{5}\begin{pmatrix}5\\6\\-3\end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{25+36+9}}\begin{pmatrix}5\\6\\-3\end{pmatrix}$$

(4) $$\operatorname{rank}\begin{pmatrix}1 & -3 & 6 & 5\\ -1 & 2 & -5 & -3\\ 1 & 1 & 2 & -3\end{pmatrix}$$

✓ ✓ $3\cdot I - II$ ✗ $-I-2II$ ✗

$$\begin{cases} a-3b=5\\ -a+2b=-3\\ a+b=-3\end{cases}$$

$b=-2$ $a=-1$

$\Rightarrow \operatorname{rank}=2$

(5) $$\begin{cases} x+y-z=3 & ✓\\ x-z+4w=-5 & ✓\\ x+w=-1 & ✓\\ y-2z+2w=0 & ✓\end{cases}$$

$$\begin{cases} z=x+y-3\\ w=-x-1\\ x-x-y+3-4x-4=-5\\ y-2x-2y+6-2x-2=0\end{cases}$$

$$\begin{cases} +4x+y=+4\\ +4x+y=+4\\ z=\dots\\ w=\dots\end{cases}$$

$$\begin{cases} x=t\\ y=4-4t\\ z=1-3t\\ w=-t-1\end{cases} \Leftrightarrow \underline{\underline{\begin{pmatrix}0\\4\\1\\-1\end{pmatrix}}} + \operatorname{Span}\underline{\underline{\begin{pmatrix}1\\-4\\-3\\-1\end{pmatrix}}}$$

Verifica: $0+4-1=3$ ok $\quad -1+4-3=0$ ok

$0-1-4=-5$ ok $\quad -1-3+4=0$ ok

⑥ $\begin{pmatrix} 2 & 1 \\ 7 & -4 \end{pmatrix}$

$\lambda_1+\lambda_2 = tr = -2$

$\lambda_1 \cdot \lambda_2 = det = -15$

$\lambda_1 = -5 \qquad \lambda_2 = 3$

$V_1: \begin{cases} 2x+y=-5x \\ 7x-4y=-5y \end{cases} \qquad V_2: \begin{cases} 2x+y=3x \\ \end{cases}$

$7x+y=0$

$V_1 = \begin{pmatrix} -1 \\ 7 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ESE $f: \mathbb{R}^3 \to \mathbb{R}^3$ associata a $\begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix}$.

Ⓐ $f\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ -10 \end{pmatrix}$

Ⓑ dim $Ker(f) / Im(f)$

$\dim(Im(f)) = rank \begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix} = 2$

✓ ✓ $-I-II$ ✗

$\dim(Ker(f)) = 3-2 = 1$

$$\mathrm{Ker}(f):\ \begin{cases} \cancel{4x - 2y - 2z = 0} \\ 2x - y - z = 0 \ \checkmark \\ \quad\ \ y - z = 0 \ \checkmark \\ -x \quad\ + z = 0 \ \checkmark \end{cases} \qquad \begin{cases} y = z \\ x = z \end{cases} \qquad \mathrm{Ker}(f) = \mathrm{Span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

ⓒ $p_A(t)$

$$= \det(t \cdot I_3 - A) = \det \begin{pmatrix} t-4 & 2 & 2 \\ 0 & t-1 & 1 \\ 2 & 0 & t-2 \end{pmatrix}$$

$$= \det \begin{pmatrix} t-4 & 2 & 0 \\ 0 & t-1 & 2-t \\ 2 & 0 & t-2 \end{pmatrix}$$

$$= (t-2) \cdot \det \begin{pmatrix} t-4 & 2 & 0 \\ 0 & t-1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= (t-2) \cdot \det \begin{pmatrix} t-4 & 2 & 0 \\ 0 & t-1 & -1 \\ 2 & t-1 & 0 \end{pmatrix}$$

$$= (t-2) \cdot \left(-(-1) \cdot (t^2 - 5t + 4 - 4)\right)$$

$$= t(t-2)(t-5)$$

$$= t^3 - 7t^2 + 10t$$

ⓓ $\lambda_1 = 0 \qquad \lambda_2 = 2 \qquad \lambda_3 = 5$

distinti $\Longrightarrow$ è diagonalizzabile.

(E) Trovare M t.c. $M^{-1} \cdot A \cdot M$ sia diagonale.

M è la matr. che ha come colonne gli autovettori:

$$A = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix}$$

$\lambda_1 = 0 \qquad \lambda_2 = 2 \qquad \lambda_3 = 5$

$v_1: \begin{cases} 4x-2y-2z=0 \\ y-z=0 \\ -2x+2z=0 \end{cases} \qquad v_2: \begin{cases} \overline{\qquad\qquad} \\ y-z=2y \\ -2x+2z=2z \end{cases} \qquad v_3: \begin{cases} \overline{\qquad\qquad} \\ y-z=5y \\ -2x+2z=5z \end{cases}$

$\begin{cases} y=z \\ x=z \\ \overline{\qquad\qquad} \end{cases} \qquad \begin{matrix} x=0 \\ z=-y \end{matrix} \qquad \begin{matrix} 4y+z=0 \\ 2x+3z=0 \end{matrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix}$

$$M = \begin{pmatrix} 1 & 0 & 6 \\ 1 & 1 & 1 \\ 1 & -1 & -4 \end{pmatrix}$$

Prova 2

(3) $\det \begin{pmatrix} 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & -3 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}$

$$= \det \begin{pmatrix} 0 & -1 & 0 & 0 \\ 3 & 1 & 1 & 3 \\ -6 & 3 & -1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}$$

$$= -(-1)\cdot\det\begin{pmatrix} 3 & 1 & 3 \\ -6 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= 3\cdot\det\begin{pmatrix} 1 & 1 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3\cdot(-1+0+0+3+2+0) = 12$$

4) Trovare base di $\mathbb{R}^4$ che contenga

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \\ 3 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \\ 1 \\ -3 \end{pmatrix}}$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

5) $W = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right)$; base $W^\perp$

$u, v \leadsto u\times v$ è $\perp$ a entrambi, cioè a $\operatorname{Span}(u,v)$

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\wedge\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

ESE

$$R : \begin{cases} 4x+y+3z+2w=0 \\ 2x-3y+5z+w=0 \end{cases}$$

(A) trovare equaz. param.

- $$\begin{cases} y=-4x-3z-2w \\ 2x+12x+9z+6w+5z+w=0 \end{cases}$$

$$\begin{cases} y=-4x-3z-2w \\ 14x+14z+7w=0 \end{cases} \qquad \begin{cases} w=-2x-2z \\ y=-4x-3z \\ \quad +4x+4z = z \end{cases}$$

$$\begin{cases} x=x \\ y=z \\ z=z \\ w=-2x-2z \end{cases} \longleftrightarrow \begin{cases} x=t \\ y=s \\ z=s \\ w=-2t-2s \end{cases} \longleftrightarrow \mathrm{Span}\begin{pmatrix}1\\0\\0\\-2\end{pmatrix}\begin{pmatrix}0\\1\\1\\-2\end{pmatrix}$$

$$R : \begin{cases} 4x+y+3z+2w=0 \\ 2x-3y+5z+w=0 \end{cases} \qquad \dim R = 4-2=2$$

Posso cercare un generatore con $w=0$ e le altre

$$\begin{cases} 4x+y+3z=0 \\ 2x-3y+5z=0 \end{cases} ; \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\1\\3\end{pmatrix}\times\begin{pmatrix}2\\-3\\5\end{pmatrix}=\begin{pmatrix}14\\-14\\-14\end{pmatrix}\longrightarrow\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}$$

Posso cercare un altro con $x=0$ e le altre

$$\begin{cases} y+3z+2w=0 \\ -3y+5z+w=0 \end{cases} \qquad \begin{pmatrix}y\\z\\w\end{pmatrix}=\begin{pmatrix}1\\3\\2\end{pmatrix}\times\begin{pmatrix}-3\\5\\1\end{pmatrix}=\begin{pmatrix}-7\\-7\\14\end{pmatrix}\to\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\1\\-2\end{pmatrix}$$

$$S = \mathrm{Span}\begin{pmatrix}2\\1\\-1\\3\end{pmatrix}\begin{pmatrix}-1\\3\\1\\5\end{pmatrix}$$

(B) equaz. cart. di S

• $S: \begin{cases} x = 2t - s \\ y = t + 3s \\ z = -t + s \\ w = 3t + 5s \end{cases}$

$$\begin{cases} \mathrm{I} & s = 2t - x \\ \mathrm{II} & z = -t + 2t - x \end{cases} \quad \begin{aligned} &\to t = x + z \\ &\to s = x + 2z \end{aligned}$$

$$\begin{cases} y = x + z + 3x + 6z \\ w = 3x + 3z + 5x + 10z \end{cases} \qquad \boxed{\begin{cases} 4x - y + 7z = 0 \\ 8x + 13z - w = 0 \end{cases}}$$

• $\dim S = 2$: scrivo $4 - 2 = 2$ equaz.

posso cercare una in cui non compare w

$$\begin{pmatrix}2\\1\\-1\end{pmatrix} \times \begin{pmatrix}-1\\3\\1\end{pmatrix} = \begin{pmatrix}4\\-1\\7\end{pmatrix}$$

$$\boxed{4x - y + 7 \cdot z + 0 \cdot w = 0}$$

posso cercare altra in cui non compare x

$$\begin{pmatrix}1\\-1\\3\end{pmatrix} \times \begin{pmatrix}3\\1\\5\end{pmatrix} = \begin{pmatrix}-8\\4\\4\end{pmatrix} \to \begin{pmatrix}-2\\1\\1\end{pmatrix}$$

$$\boxed{0 \cdot x - 2y + z + w = 0}$$

(C) Trovare base di $R \cap S$

$$R: \begin{cases} 4x + y + 3z + 2w = 0 \\ 2x - 3y + 5z + w = 0 \end{cases} \qquad S = \mathrm{Span}\underbrace{\begin{pmatrix}2\\1\\-1\\3\end{pmatrix}\begin{pmatrix}-1\\3\\1\\5\end{pmatrix}}_{\begin{pmatrix}2t - s\\ t + 3s\\ -t + s\\ 3t + 5s\end{pmatrix}}$$

$$\begin{cases} 4(2t-s)+(t+3s)+3(-t+1)+2(3t+5s)=0 \\ 2(\;''\;)-3(\;''\;)+5(\;''\;)+1(\;''\;)=0 \end{cases}$$

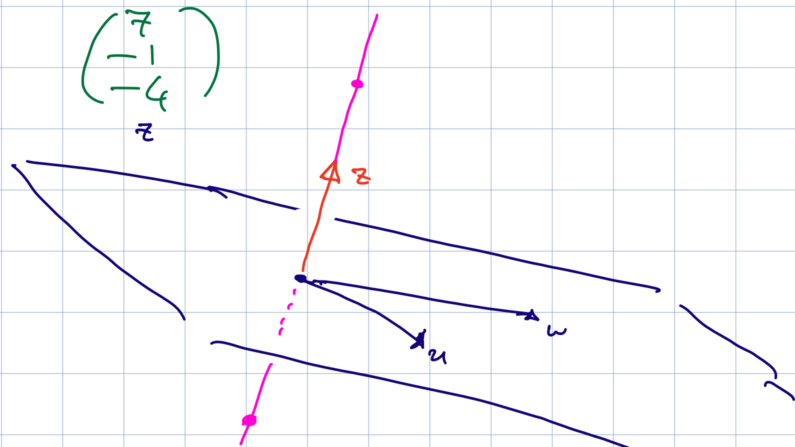
$t+1=0$ $\qquad \begin{pmatrix} 3 \\ -2 \\ -2 \\ -2 \end{pmatrix}$

D) $\dim(R+S) = \dim(R)+\dim(S)-\dim(R\cap S)$
$= 2 + 2 - 1 = 3$

Prova 3 (12/5/24):

3) Trovare <u>tutti</u> i v, $\|v\|=1$, $v \perp \operatorname{Span}\left(\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right)$

$$\underbrace{\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}}_{u} \times \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{w} = \underbrace{\begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}}_{z}$$



$$\pm\frac{1}{\sqrt{66}}\begin{pmatrix} -7 \\ 1 \\ 4 \end{pmatrix}$$

4) $\dim\left(\operatorname{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 4 \\ 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ -2 \\ 2 \end{pmatrix}\right)\right)$

$\checkmark \quad \checkmark \quad -I+2II \quad 3\cdot I-II$

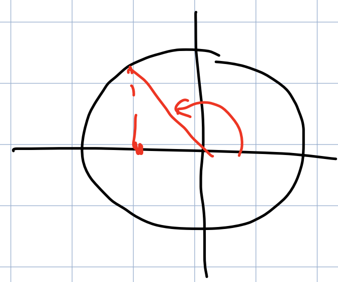
$\times \quad \times$

$= 2$

⑤ angolo tra $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ e $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$$\cos(\measuredangle(u,v)) = \frac{u \cdot v}{\|u\| \cdot \|v\|}$$

$$\cos = \frac{-3}{3 \cdot \sqrt{2}} = -\frac{1}{\sqrt{2}}$$



$$\vartheta = \frac{3}{4}\pi$$

ESE

$$\left(\begin{array}{ccc|c} h & -1 & 0 & k \\ 2 & 1 & -1 & 0 \\ -3 & h & -2h & 0 \end{array}\right)$$

(A) Quante soluz?

Se $\det(\text{incompl } 3\times3) \neq 0$ allora unica.

$$\begin{aligned} \det &= -2h^2 - 3 + 0 + 0 - 4h + h^2 \\ &= -h^2 - 4h - 3 \\ &= -(h^2 + 4h + 3) \\ &= -(h+1)(h+3) \end{aligned}$$

Se $h \neq -1$ e $h \neq -3$ soluz. unica.

$$h=-1 \qquad \begin{array}{ccc|c} -1 & -1 & 0 & k \\ 2 & 1 & -1 & 0 \\ \hline \end{array}$$

$$-I-2\cdot II \quad \begin{array}{ccc|c} -3 & -1 & 2 & 0 \end{array} \quad -k-2\cdot II$$

infinite soluz. per $k=0$
nessuna per $k \neq 0$

$$h=-3 \qquad \begin{array}{ccc|c} -3 & -1 & 0 & k \\ 2 & 1 & -1 & 0 \end{array}$$

$$-3\cdot I-6\cdot II \quad \begin{array}{ccc|c} -3 & -3 & 6 & 0 \end{array} \quad -3\cdot k-6\cdot 0$$

infinite soluz. per $k=0$
nessuna per $k \neq 0$

(B) $\begin{cases} x+y=0 \\ 2x+y-z=0 \end{cases}$ $\qquad \begin{cases} x+y=0 \\ x-z=0 \end{cases}$ $\qquad \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(C) $W_2 = \text{Span} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Base W_1+W_2

$\dim W_1 = 1$
$\dim W_2 = 2$

	W_1		W_2		W_1+W_2		$W_1 \cap W_2$
A	1	+	2	=	2	+	1
	1	+	2	=	3	+	0

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$