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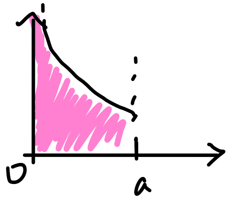
ISTITUZIONI DI MATEMATICA I 29/02/2024

Programma di oggi:

I. $\exists$ e $\nexists$ di int. impropri – criteri di confronto
II. Esercizi sugli integrali generalizzati
III. Esercizi sugli integrali

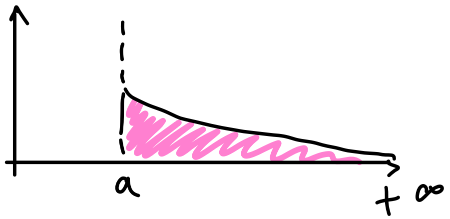
I Criteri di confronto per $\exists$ e $\nexists$ di integrali impropri

Abbiamo visto che



← qualsiasi $a>0$, $a \in \mathbb{R}$

$$\int_0^a \frac{1}{x^\alpha}\,dx \quad \exists \text{ finito} \iff \alpha<1$$



$$\int_a^{+\infty} \frac{1}{x^\alpha}\,dx \quad \exists \text{ finito} \iff \alpha>1$$

← qualsiasi $a>0$, $a \in \mathbb{R}$

$\frac{1}{x^\alpha}$ è il prototipo di funzione integrabile / non integrabile in senso generalizzato vicino a 0 o all'∞.

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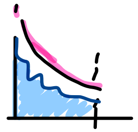
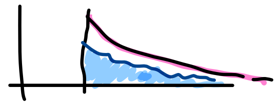
Per confronto:

- Se $|f| \leq \frac{1}{x^\alpha}$ con $\alpha < 1 \Rightarrow$ f int. in senso gen. vicino a 0



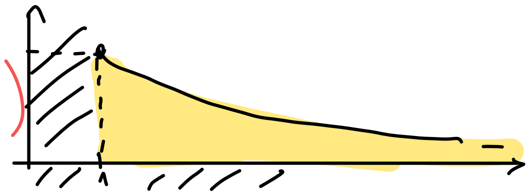
- Se $|f| \leq \frac{1}{x^\alpha}$ con $\alpha > 1 \Rightarrow$ f int. in senso gen. a $+\infty$



- se $|f| \leq g$ con g int. in senso gen. su $(a,b) \Rightarrow$ f int. in senso gen. su (a,b)

CRITERIO DI CONFRONTO SERIE - INTEGRALE ALL'∞:

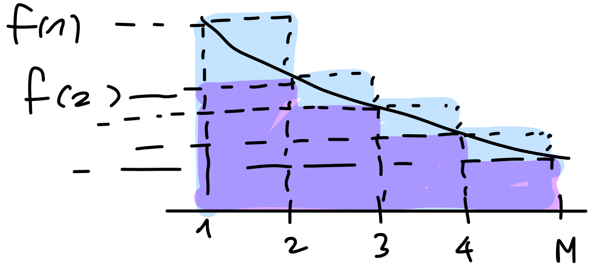
Sia $f: [1, +\infty) \longrightarrow \mathbb{R}$, $f \geq 0$, f decrescente (es. $\frac{1}{x^\alpha}$)



oss. $$\int_1^{+\infty} f(x)\,dx = \lim_{M \to +\infty} \int_1^M f(x)\,dx \in \mathbb{R} \cup \{+\infty\}$$

Stimiamo dall'alto e dal basso l'integrale: sia $M \in \mathbb{N}$

$$\sum_{n=2}^{M} f(n) \cdot 1 \leq \int_1^M f(x)\,dx \leq \sum_{n=1}^{M-1} f(n) \cdot 1$$



Passando al limite per $M \to +\infty$

$$\sum_{n \geq 2} f(n) \leq \int_1^{+\infty} f(x)\,dx \leq \sum_{n \geq 1} f(n)$$

$\Rightarrow$ Serie e integrale hanno lo stesso carattere (cioè o convergono entrambi o divergono entrambi).

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OSS $\sum_{n\geq 1} \frac{1}{\sqrt{n}}, \sum_{n\geq 1} \frac{1}{n} = +\infty$, $\sum_{n\geq 1} \frac{1}{n^\alpha} = +\infty \quad \forall \alpha \leq 1$

$\sum_{n\geq 1} \frac{1}{n^2}, \sum_{n\geq 1} \frac{1}{n^3} \in \mathbb{R}$, $\sum_{n\geq 1} \frac{1}{n^\alpha} \in \mathbb{R} \quad \forall \alpha > 1$

II Esercizi int. generalizzati

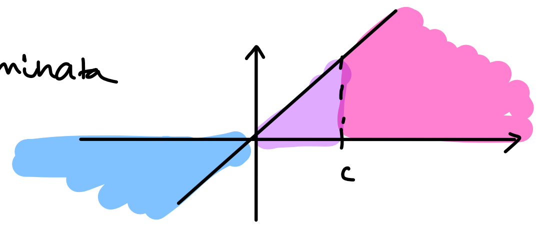
$$\int_1^{+\infty} x\,dx = +\infty$$

$$= \lim_{M\to+\infty} \int_1^M x\,dx = \lim_{M\to+\infty} \frac{x^2}{2}\Big|_1^M = \lim_{M\to+\infty} \left[\frac{M^2}{2} - \frac{1}{2}\right] = +\infty.$$

Scrittura alternativa (formale): $\int_1^{+\infty} x\,dx = \left[\frac{x^2}{2}\Big|_1^{+\infty}\right] = \frac{(+\infty)^2}{2} - \frac{1}{2} = +\infty.$

$$\int_{-\infty}^{+\infty} x\,dx \quad \nexists$$

perché $\int_{-\infty}^{+\infty} x\,dx \underset{\text{per def.}}{=} \int_{-\infty}^{c} x\,dx + \int_c^{+\infty} x\,dx = -\infty + \infty$ forma indeterminata



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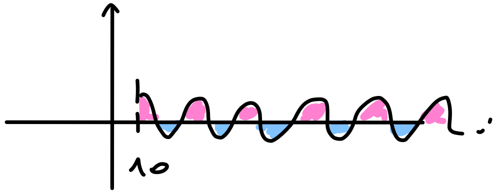
$$\int_{10}^{+\infty} \sin x \, dx \quad \nexists$$



$$\int \sin x \, dx = -\cos x + c$$

$$\Rightarrow \int_{10}^{+\infty} \sin x \, dx = -\cos(+\infty) + \cos(10) \quad \nexists \text{ perché } \nexists \lim_{M \to +\infty} \cos(M).$$

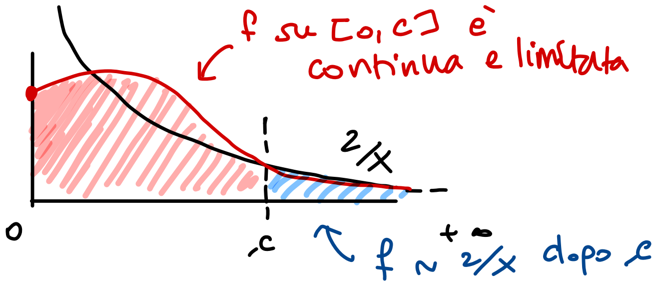
$$\int_{0}^{+\infty} \frac{2x+1}{1+x^2} \, dx = +\infty$$

oss: si tratta di un int. gen. perché l'intervallo di integrazione è illimitato (estremo destro $= +\infty$), la funzione è invece continua e limitata.

oss: l'integrando è $>0 \Rightarrow \int_0^{+\infty} f(x)\,dx \in \mathbb{R}^+ \cup \{+\infty\}$.

oss: per valutare se l'integrale $\exists$ o è $+\infty$, usiamo il confronto con $\frac{1}{x^\alpha}$ per x grande, $x > c$ per qualche $c>0$:

$$\frac{2x+1}{1+x^2} \sim \frac{2x}{x^2} = \frac{2}{x} \Rightarrow$$



$$\Rightarrow \int_0^{+\infty} f(x)\,dx = \underbrace{\int_0^c f(x)\,dx}_{\in \mathbb{R}} + \underbrace{\int_c^{+\infty} f(x)\,dx}_{} \sim \int_c^{+\infty} \frac{1}{x}\,dx = +\infty$$

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Conto: $\displaystyle\int_0^{+\infty} \frac{2x+1}{1+x^2}\,dx = \int_0^{+\infty} \frac{2x}{1+x^2}\,dx + \int_0^{+\infty} \frac{1}{1+x^2}\,dx = \Big[\ln(1+x^2) + \arctan(x)\Big|_0^{+\infty}$

$$= \ln(+\infty) + \underbrace{\arctan(+\infty)}_{=\frac{\pi}{2}} - \underbrace{\ln(1)}_{=0} - \underbrace{\arctan(0)}_{=0} = +\infty + \frac{\pi}{2} = +\infty.$$

$$\int_1^{+\infty} e^{-3x}\,dx = \frac{1}{3e^3}$$

oss: l'int. generalizzato $\exists$ per confronto con $\frac{1}{x^3}$ (o $\frac{1}{x^\alpha}$, $\alpha>1$ qualsiasi).

Infatti: $\exists\, c \gg 1$ (cioè c grande) tc $x^3 < e^{3x}$

$$\Rightarrow e^{-3x} < \frac{1}{x^3} \quad \forall x > c \Rightarrow \int_1^{+\infty} f(x)\,dx = \int_1^{c} f(x)\,dx + \int_c^{+\infty} f(x) \leq \underbrace{\int_1^{c} f(x)\,dx}_{\in\mathbb{R}} + \underbrace{\int_c^{+\infty} \frac{1}{x^3}\,dx}_{\in\mathbb{R}}$$

(f continua su $[1, c]$) $\quad (\alpha > 1)$

Conto:

$$\int_1^{+\infty} e^{-3x}\,dx = \Big[\frac{e^{-3x}}{-3}\Big|_1^{+\infty} = \frac{e^{-\infty}}{-3} - \frac{e^{-3}}{-3} = \frac{e^{-3}}{3}$$

$e^{-\infty} = \frac{1}{e^{+\infty}} = \frac{1}{+\infty} = 0^+.$

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$$\int_{-\infty}^{+\infty} x^2 e^{-|x|}\,dx = 4$$

Conto:

$$\int \cdots = 2\int_0^{+\infty} x^2 e^{-|x|}\,dx = 2\int_0^{+\infty} x^2 e^{-x}\,dx = 2\cdot 2 = 4$$

(★) Lo calcoliamo per parti

- f pari, ≥ 0 (non rischiamo di avere forme indeterminate $\infty - \infty$)

oppure, con un conto diretto:

- $$\int_{-\infty}^{0} x^2 e^{-|x|}\,dx = \int_{+\infty}^{0} t^2 e^{-t}(-dt) = \int_0^{+\infty} t^2 e^{-t}\,dt$$

$t = -x$, $x = -t$, $dx = -dt$, $|x| = -x = t$

$x = -\infty \Rightarrow t = +\infty$, $x = 0 \Rightarrow t = 0$

(★)

$$\int f'g = fg - \int f g'$$

$$\int x^2 e^{-x}\,dx = -x^2e^{-x} + 2\int x\, e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}$$

$$= (-x^2 - 2x - 2)\,e^{-x} + C$$

$g = x^2$, $f' = e^{-x}$; $g' = 2x$, $f = -e^{-x}$

$g = x$, $f' = e^{-x}$; $g' = 1$, $f = -e^{-x}$

$$\Rightarrow \int_0^{+\infty} x^2 e^{-x}\,dx = \left[\lim_{x\to+\infty} (-x^2-2x-2)e^{-x}\right] - (-2)e^0 = 2$$

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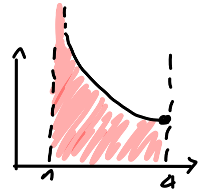
$$\int_1^4 \frac{1}{(x-1)^\alpha}\,dx = \begin{cases} +\infty & \text{se } \alpha \geq 1 \\ \frac{3^{1-\alpha}}{1-\alpha} & \text{se } \alpha < 1 \end{cases}$$

oss: $f(x) = \frac{1}{(x-1)^\alpha}$



è una funzione continua in $(1,4]$, con un asintoto verticale in 1

Conto: $\int_1^4 \frac{1}{(x-1)^\alpha}\,dx = \int_0^3 \frac{1}{t^\alpha}\,dt$

$t = x-1$

$dt = dx$

$x=1 \Rightarrow t=0$
$x=4 \Rightarrow t=3$

ci siamo ricondotti al caso noto:

$$\int_0^3 \frac{1}{t^\alpha}\,dt = \begin{cases} +\infty & \text{se } \alpha \geq 1 \\ \in \mathbb{R} & \text{se } \alpha < 1 \end{cases}$$

se $\alpha < 1$

$$= \left[\frac{t^{1-\alpha}}{1-\alpha}\right]_0^3$$

$$\int_0^1 x \cdot \ln x\,dx = \frac{1}{4}$$

(ES. 72 DEL LIBRO)

oss: $f(x) := x \cdot \ln x \Rightarrow f : (0,1] \to \mathbb{R}$ continua.

$\lim_{x\to 0^+} x \ln x = 0 \Rightarrow f(x) := \begin{cases} x \ln x & \text{in } (0,1] \\ 0 & \text{se } x = 0 \end{cases}$ è continua

in $[0,1] \Rightarrow \int_0^1 f(x)dx \in \mathbb{R}$ (e non è un int. generalizzato).

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Conto: $\int \underset{f'}{x} \cdot \underset{g}{\ln x}\, dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx = \frac{x^2}{2} - \frac{x^2}{4} + C$

$\int f' \cdot g = f \cdot g - \int f \cdot g'$

$\Rightarrow \int_0^1 x \ln x \, dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\, dx = 2$$

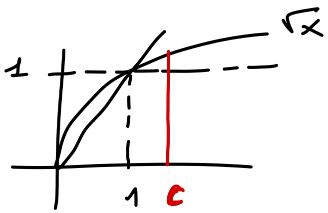
(ES. 73 DEL LIBRO)

oss: $\int_0^{+\infty} f(x)\,dx = \int_0^c f(x)\,dx + \int_c^{+\infty} f(x)\,dx$, con $c > 0$ qualsiasi.

$e^{-\sqrt{x}} \leq 1 \Rightarrow \frac{1}{\sqrt{x}} e^{-\sqrt{x}} \leq \frac{1}{\sqrt{x}} \Rightarrow \int_0^c f(x)\,dx \leq \int_0^c \frac{1}{\sqrt{x}}\,dx \in \mathbb{R}$

per integrabilità di $\frac{1}{x^\alpha}$ vicino a 0, $\alpha < 1$.

Se $c > 1 \Rightarrow \sqrt{x} \geq 1 \quad \forall x \geq c$



$\Rightarrow \frac{1}{\sqrt{x}} \leq 1 \quad \forall x \geq c$

$\Rightarrow \frac{1}{\sqrt{x}} e^{-\sqrt{x}} \leq e^{-\sqrt{x}} \quad \forall x \geq c$

$\Rightarrow \int_c^{+\infty} f(x)dx \leq \int_c^{+\infty} e^{-\sqrt{x}}\, dx \in \mathbb{R}$ per confronto con $\frac{1}{x^2}$ all'∞

Cioè $\exists d$, senza perdita di generalità $d > c$, tale che $e^{-\sqrt{x}} \leq \frac{1}{x^2}$ $\forall x \geq d$.

$$\Rightarrow \int_c^{+\infty} f(x)dx \leq \underbrace{\int_c^{d} e^{-\sqrt{x}}\, dx}_{\in \mathbb{R}} + \underbrace{\int_d^{+\infty} \frac{1}{x^2}\, dx}_{\in \mathbb{R}}$$

$\Rightarrow \int_0^{+\infty} f\, dx \in \mathbb{R}$.

Conto:
$$\frac{d}{dx}\left(e^{-\sqrt{x}}\right) = e^{-\sqrt{x}} \cdot \frac{d}{dx}\left(-\sqrt{x}\right) = e^{-\sqrt{x}}\left(-\frac{1}{2\sqrt{x}}\right) = -\frac{1}{2}\left[\frac{1}{\sqrt{x}} e^{-\sqrt{x}}\right]$$

$$\Rightarrow \frac{1}{\sqrt{x}} e^{-\sqrt{x}} = \frac{d}{dx}\left(-2e^{-\sqrt{x}}\right)$$

$$\Rightarrow \int_0^{+\infty} \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\, dx = \left[-2e^{-\sqrt{x}}\right|_0^{+\infty} = -2 \cdot e^{-\infty} + 2 \cdot e^0 = 2.$$

$$\int_0^{+\infty} e^{-2x} \sin(e^{-x})\, dx \quad \in \mathbb{R}$$

(ES. 74 DEL LIBRO. Dire se converge)

$f : [0, +\infty) \to \mathbb{R}$ continua

A $+\infty$: $\sin(e^{-x}) \in [0,1]$ perché $e^{-x} \to 0$.

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$\Rightarrow$ $e^{-2x}\sin(e^{-x}) \sim e^{-2x}$ integrabile all'∞ per un conto diretto o per confronto con $\frac{1}{x^\alpha}$, $\alpha>1$.

$$\int_0^1 \frac{\sin x}{x^2}\,dx = +\infty$$

(ES. 75 DEL LIBRO)

$f : [0,1] \to \mathbb{R}$ continua, con un asintoto verticale in 0

Per $x \sim 0 \Rightarrow \sin x \sim x \Rightarrow \frac{\sin x}{x^2} \sim \frac{1}{x}$ non int. in senso gener. vicino a 0

$$\Rightarrow \int_0^1 \frac{\sin x}{x^2}\,dx = \underbrace{\int_0^\varepsilon \frac{\sin x}{x^2}\,dx}_{=+\infty} + \underbrace{\int_\varepsilon^1 \frac{\sin x}{x^2}\,dx}_{\in \mathbb{R}} = +\infty$$

III ESERCIZI SUGLI INTEGRALI

FUNZIONI RAZIONALI

$$\int \frac{1}{x^2(x-1)}\,dx$$

La strategia è la stessa usata per il caso $\int P(x)/Q(x)$ con $Q(x)$ di grado 2.

Il denominatore ha 3 radici:

$x=0$ di molteplicità 2 $\rightsquigarrow$ Radice doppia $\rightsquigarrow$ $\frac{A}{x}+\frac{B}{x^2}$

$x=1$ di molteplicità 1 $\rightsquigarrow$ Radice semplice $\rightsquigarrow$ $c/(x-1)$

Cerchiamo $A, B, C \in \mathbb{R}$ tali che $\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{c}{x-1}$

$$\Rightarrow \quad Ax(x-1)+B(x-1)+Cx^2=1 \Rightarrow \begin{cases} A+C=0 \\ -A+B=0 \\ -B=1 \end{cases} \qquad \begin{cases} A=B=-1 \\ C=1 \end{cases}$$

$$\Rightarrow \int \frac{1}{x^2(x-1)}\,dx = -\int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{1}{x-1}\,dx = -\ln|x| + \frac{1}{x} + \ln|x-1| + \text{cost.}$$

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$$\int_1^{10} \frac{x^4-2x^3+2x+1}{x^3+x}\,dx = \underbrace{\int_1^{10}(x-2)\,dx}_{\text{facile}} + \overbrace{\int_1^{10}\frac{(-x^2+4x+1)}{x^3+x}\,dx}^{\text{forma canonica}}$$

Il grado del numer. è > grado del denominatore

$\Rightarrow$ effettuiamo una divisione tra polinomi

$$\frac{P(x)}{Q(x)} = N(x) + \frac{R(x)}{Q(x)}$$

$$\begin{array}{l|l} x^4 - 2x^3 + 2x + 1 & x^3 + x \\ \hline x^4 + x^2 & x - 2 \\ \hline -2x^3 - x^2 + 2x + 1 & \\ -2x^3 \quad - 2x & \\ \hline -x^2 + 4x + 1 & \end{array}$$

$$\int (x-2)\,dx = \frac{x^2}{2} - 2x + \text{cost.}$$

$$\frac{-x^2+4x+1}{x^3+x} = \frac{-x^2+4x+1}{x(1+x^2)}$$

usiamo la stessa strategia usata per Q di grado 2:

Radice semplice $x=0$ $\rightsquigarrow \dfrac{A}{x}$

Radici complesse coniugate $\rightsquigarrow \dfrac{Bx}{1+x^2} + \dfrac{C}{1+x^2}$

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Cerchiamo $A, B, C \in \mathbb{R}$ tali che $\dfrac{-x^2+4x+1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx}{1+x^2} + \dfrac{C}{1+x^2}$

$$\Rightarrow \quad -x^2+4x+1 = A(1+x^2) + Bx^2 + Cx$$

$$\Rightarrow \begin{cases} -1 = A+B \\ 4 = C \\ 1 = A \end{cases} \quad \Rightarrow \quad A=1,\ B=-2,\ C=4$$

$$\Rightarrow \int \frac{-x^2+4x+1}{x(1+x^2)}\,dx = \int \frac{1}{x}\,dx - \int \frac{2x}{1+x^2}\,dx + 4\int \frac{1}{1+x^2}\,dx$$

$$= \ln|x| - \ln(1+x^2) + 4\arctan(x) + \text{cost.}$$

$$\Rightarrow \int_1^{10} \dots\, dx = \left[\frac{x^2}{2} - 2x + \ln|x| - \ln(1+x^2) + 4\arctan x \right]_1^{10}$$

$$= 50 - 20 + \ln(10) - \ln(101) + 4\arctan 10 +$$

$$- \frac{1}{2} + 2 - \cancel{\ln 1} + \ln 2 - 4\underbrace{\arctan 1}_{= \pi/4}$$

$$= \boxed{\frac{63}{2} + \ln\left(\frac{5}{101}\right) + 4\arctan(10) - \pi.}$$

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PER PARTI

$$\int x^2 \ln x \, dx$$

Per parti $\int f' \cdot g = fg - \int f \cdot g'$

$x^2 = f' \rightsquigarrow f = \frac{x^3}{3}$

$\ln x = g \rightsquigarrow g' = \frac{1}{x}$

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \cdot \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx = \frac{x^3}{3} \cdot \ln x - \frac{x^3}{9} + c = \frac{x^3}{9}\left[3 \ln x - 1\right] + c.$$

$$\int e^x \cos(2x) \, dx$$

$f'(x) = e^x \rightsquigarrow f(x) = e^x$

$g(x) = \cos(2x) \rightsquigarrow g'(x) = -2\sin(2x)$

$$\int \cdots = e^x \cos(2x) + 2\int e^x \sin(2x) = e^x\left(\cos(2x) + 2\sin(2x)\right) - 4\int e^x \cos(2x) dx$$

$f'(x) = e^x \rightsquigarrow f(x) = e^x$

$g(x) = \sin(2x) \rightsquigarrow g'(x) = 2\cos(2x)$

$$\Rightarrow \quad 5\int e^x \cos(2x) \, dx = e^x\left(\cos(2x) + 2\sin(2x)\right) + \text{cost.}$$

$$\Rightarrow \quad \int e^x \cos(2x) \, dx = \frac{e^x}{5}\left(\cos(2x) + 2\sin(2x)\right) + \text{cost.}$$

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$$\int_2^3 \frac{\ln(\ln x)}{x}\,dx$$

Per parti: $\int f' \cdot g = f \cdot g - \int f \cdot g'$

$f' = \frac{1}{x} \rightsquigarrow f(x) = \ln x$

$g = \ln(\ln x) \rightsquigarrow g'(x) = \frac{1}{\ln x} \cdot (\ln x)' = \frac{1}{\ln x} \cdot \frac{1}{x}$

$$\Rightarrow \int_2^3 \cdots = \left[\ln x \cdot \ln(\ln x)\right|_2^3 - \int_2^3 \underbrace{\ln x \cdot \frac{1}{\ln x}}_{=1} \cdot \frac{1}{x}\,dx$$

$$= \left[\ln x \cdot \ln(\ln x) - \ln|x|\right|_2^3$$

$$= \ln 3 \cdot \ln(\ln 3) - \ln 3 - \ln 2 \cdot \ln(\ln 2) + \ln 2$$

PER SOSTITUZIONE

$$\int \cos^2 x \cdot \sin x\,dx$$

Usiamo $\int f(g) \cdot g' = F(g)$

Qui usiamo $f(t) = t^2$,

$g(x) = \cos x \Rightarrow g'(x) = -\sin x$

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$$\Rightarrow \int \cos^2 x \cdot \sin x \, dx = - \int \cos^2 x \cdot (-\sin x) \, dx = -\frac{\cos^3 x}{3} + \text{cost.}$$

Oppure: $y = \cos x \qquad dy = -\sin x \, dx$

$$\int \cos^2 x \cdot \sin x \, dx = -\int y^2 \, dy = -\frac{y^3}{3} + \text{Cost.} = -\frac{\cos^3 x}{3} + \text{cost}$$

OSS $\displaystyle \int \cos(2x) \sin x \, dx = \int (2\cos^2 x - 1) \sin x \, dx = 2\left(-\frac{\cos^3 x}{3}\right) + \cos x + c.$

$$\int \frac{1}{x+3+2\sqrt{x-1}} \, dx$$

$$\int \frac{1}{x+3+2\sqrt{x-1}} \boxed{dx} \longleftarrow \begin{cases} \boxed{t = \sqrt{x-1}} \Rightarrow t^2 = x-1 \\ \Rightarrow x = t^2+1 \Rightarrow \boxed{dx = 2t\,dt} \end{cases}$$

$$= \int \frac{1}{t^2+2t+4} \boxed{2t\,dt}$$

$\left\{\begin{array}{l} \text{Il denominatore ha radici complesse coniugate} \\ \displaystyle \Rightarrow \int \frac{ax+b}{(cx+d)^2+e^2} \, dx = \frac{A(cx+d)}{(cx+d)^2+e^2} + \frac{B}{(cx+d)^2+e^2} \end{array}\right.$

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$$= \int \frac{2t}{(t+1)^2+3}\,dt = \frac{2}{3}\int \frac{t}{\left(\frac{t+1}{\sqrt{3}}\right)^2+1}\,\boxed{dt} \leftarrow \left\{ \begin{array}{l} \boxed{s = \frac{t+1}{\sqrt{3}}} \\ t = \sqrt{3}\,s - 1 \\ \boxed{dt = \sqrt{3}\,ds} \end{array} \right.$$

$$= \frac{2}{3}\int \frac{\sqrt{3}\,s-1}{s^2+1}\,\boxed{\sqrt{3}\,ds} = \int \frac{2s}{1+s^2}\,ds - \frac{2}{\sqrt{3}}\int \frac{1}{1+s^2}\,ds$$

$$= \ln(1+s^2) - \frac{2}{\sqrt{3}}\arctan(s) + c \leftarrow \left\{ \boxed{s = \frac{t+1}{\sqrt{3}} = \frac{\sqrt{x-1}+1}{\sqrt{3}}} \right.$$

$$= \ln\left(1 + \frac{(\sqrt{x-1}+1)^2}{3}\right) - \frac{2}{\sqrt{3}}\arctan\left(\frac{\sqrt{x-1}+1}{\sqrt{3}}\right) + c.$$

OSS: $\ln\left(1+\frac{(\sqrt{x-1}+1)^2}{3}\right) = \ln\left(3 + (x-\cancel{1}) + \cancel{1} + 2\sqrt{x-1}\right) - \ln 3$

$= \ln\left(x+3+2\sqrt{x-1}\right) + \text{-costante.}$

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OSS.
- $1 - x^2 = \cos^2 t$ se $x = \sin t$
- $1 + x^2 = \cosh^2 t$ se $x = \sinh t$

$$\begin{cases} \sinh t = \dfrac{e^t - e^{-t}}{2} \\ \cosh t = \dfrac{e^t + e^{-t}}{2} \end{cases} \qquad \begin{cases} (\cosh t)' = \sinh t \\ (\sinh t)' = \cosh t \end{cases}$$

$$\int \frac{x^3}{\sqrt{2-x^2}}\,dx$$ (ES. 48 DEL LIBRO)

$x^2 = 2\sin^2 t \qquad x = \sqrt{2}\sin t \qquad \sqrt{2-x^2} = \sqrt{2}\cos t \qquad \boxed{dx = \sqrt{2}\cos t\, dt}$

$$\int f(x)\boxed{dx} = \int \frac{2\sqrt{2}(\sin t)^3}{\sqrt{2}\cos t}\boxed{\sqrt{2}\cos t\, dt} = 2\sqrt{2}\int \sin^3 t\, dt = 2\sqrt{2}\int \sin^2 t \cdot \sin t\, dt$$

$$= 2\sqrt{2}\int (1-\cos^2 t)\sin t\, dt = 2\sqrt{2}\left[\int \sin t\, dt + \int \cos^2 t(-\sin t)\,dt\right]$$

$$= 2\sqrt{2}\left[-\cos t + \frac{\cos^3 t}{3} + c'\right] = -2(\sqrt{2}\cos t) + \frac{1}{3}(\sqrt{2}\cos t)^3 + c$$

$$= -2\sqrt{2-x^2} + \frac{1}{3}\sqrt{2-x^2}(2-x^2) + c = \frac{\sqrt{2-x^2}}{3}(2-x^2-6) + c$$

$$= -\frac{1}{3}\cdot\sqrt{x-1}\cdot(4+x^2) + c.$$

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$$\int_0^1 \frac{x}{\sqrt{9-x^2}}\,dx$$

1° modo: $\frac{x}{\sqrt{9-x^2}} = -\frac{(-2x)}{2\sqrt{9-x^2}} = -\frac{d}{dx}\left(\sqrt{9-x^2}\right)$

$\Rightarrow \int_0^1 \dots = -\sqrt{9-x^2}\Big|_0^1 = -\sqrt{8}+\sqrt{9} = 3-\sqrt{8}$

2° modo: $x = 3\sin t$ $\quad \boxed{dx = 3\cos t\, dt}$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2 t} = 3\cos t$

$\int \frac{x}{\sqrt{9-x^2}}\,dx = \int \frac{3\sin t}{3\cos t}\,\boxed{3\cos t\,dt} = 3\int \sin t\,dt = -3\cos t + C = -\sqrt{9-x^2} + C.$

$\rightarrow \int_0^1 \frac{x}{\sqrt{9-x^2}}\,dx = 3-\sqrt{8}.$

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$$\int_0^1 \frac{x^2+3x+1}{\sqrt{1+x^2}}\,dx$$

$x = \sinh t$ $\quad \boxed{dx = (\cosh t)\,dt} \quad \sqrt{1+x^2} = \cosh t$

$t = \operatorname{arcsinh} x \qquad x=0 \Rightarrow t = \operatorname{arcsinh}(0)=0$

$x=1 \Rightarrow t = \operatorname{arcsinh}(1) =: \beta$

$$\int_0^1 \frac{x^2+3x+1}{\sqrt{1+x^2}}\,\boxed{dx} = \int_0^\beta \frac{(\sinh t)^2+3\sinh t+1}{\cosh t}\cdot \boxed{\cosh t\,dt}$$

$$= \int_0^\beta \left[\left(\frac{e^t-e^{-t}}{2}\right)^2 + 3\left(\frac{e^t-e^{-t}}{2}\right)+1\right]dt$$

$$= \int_0^\beta \left[\frac{e^{2t}+e^{-2t}-2}{4} + \frac{3}{2}(e^t-e^{-t})+1\right]dt$$

$$= \frac{1}{4}\left[\frac{e^{2t}}{2} - \frac{e^{-2t}}{2} - 2t\right|_0^\beta + \frac{3}{2}\left[e^t+e^{-t}\right|_0^\beta + \left[t\right|_0^\beta$$

$$= \frac{\sinh(2\beta)}{4} - \frac{\beta}{2} + 3\cosh(\beta) - 3 + \beta = \frac{\sinh(2\beta)}{4} + 3\cosh(\beta) + \frac{\beta}{2} - 3$$