Ist. Mat. I – C/A

23/2/23

(1) $$\int \frac{x^3}{1+x^2}\,dx = \int \frac{x^3+x-x}{1+x^2} = \int x - \int \frac{x}{1+x^2}$$

$$= \frac{x^2}{2} \ \frac{1}{2}\log(1+x^2) + c$$

(2) $$\int \frac{2x-1}{x^2+2x+4} = \int \frac{2x-1}{(x+1)^2+3}$$

$$= \int \left(\frac{2(x+1)}{(x+1)^2+3} - \frac{3}{(x+1)^2+3} \right)$$

$$= \log(x^2+2x+4) - \int \frac{1}{\left(\frac{x+1}{\sqrt{3}}\right)^2 + 1}$$

$$= \log(x^2+2x+4) - \sqrt{3}\cdot \arctan\left(\frac{x+1}{\sqrt{3}}\right) + c$$

(3) $$\int \frac{x+2}{x^2+4x+3} = \int \frac{x+2}{(x+1)(x+3)} = \int \frac{a}{x+1} + \frac{b}{x+3} =$$

$$\begin{cases} a+b=1 \\ 3a+b=2 \end{cases} \qquad \begin{cases} a=\frac{1}{2} \\ b=\frac{1}{2} \end{cases} \qquad = \frac{1}{2}\int \frac{1}{x+1} + \frac{1}{x+3}$$

$$= \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x+3)$$

$$= \frac{1}{2}\log\left(x^2+4x+3\right)$$

$$\frac{1}{2}\int \frac{2x+4}{x^2+4x+3} = \frac{1}{2}\int \frac{(x^2+4x+3)'}{x^2+4x+3} = \frac{1}{2}\log(x^2+4x+3)+c$$

④ $$\int \frac{x+5}{x^2(x+1)} = \int \frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+1} = \cdots$$

$$\begin{aligned} &a(x^2+x) \\ +\,&b(x+1) \\ +\,&c(x^2) \end{aligned} \qquad \begin{cases} a+c=0 \\ a+b=1 \\ b=5 \end{cases} \qquad \begin{cases} b=5 \\ a=-4 \\ c=4 \end{cases}$$

$$\cdots = \int -\frac{4}{x}+\frac{5}{x^2}+\frac{4}{x+1} = -4\log|x|-\frac{5}{x}+4\cdot\log|x+1|$$

$$= 4\log\left|\frac{x+1}{x}\right| - \frac{5}{x}$$

⑤ $$\int \frac{1+2x^2}{x^4-1} = \int \frac{1+2x^2}{(x^2+1)(x^2-1)} = \int \frac{1+2x^2}{(x^2+1)(x+1)(x-1)}$$

$$= \int \frac{a}{x+1}+\frac{b}{x-1}+\frac{cx+d}{x^2+1} = \cdots$$

$$
\begin{aligned}
& a\,(x^3 - x^2 + x - 1) \\
+\ & b\,(x^3 + x^2 + x + 1) \\
+\ & c\,(x^3 \quad - x) \\
+\ & d\,(\quad x^2 \quad - 1) = 2x^2 + 1
\end{aligned}
$$

$$
\begin{cases}
a + b + c = 0 \\
-a + b + d = 2 \\
a + b - c = 0 \\
-a + b - d = 1
\end{cases}
\qquad
\begin{cases}
a + b = 0 \\
c = 0 \\
-2a + 2b = 3 \\
-a + b - d = 1
\end{cases}
\qquad
\begin{cases}
c = 0 \\
b = -a \\
-4a = 3 \\
d = -a + b - 1
\end{cases}
$$

$$
\begin{cases}
c = 0 \\
a = -3/4 \\
b = 3/4 \\
d = 1/2
\end{cases}
$$

$$
\ldots = \int -\frac{3}{4}\,\frac{1}{x+1} + \frac{3}{4}\,\frac{1}{x-1} + \frac{1}{2}\,\frac{1}{x^2+1}
$$

$$
= -\frac{3}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\arctan(x)
$$

⑥ $$\int \frac{1-2x}{(x+1)^2(x^2+1)} = \int \left(\frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{cx+d}{x^2+1} \right) = \ldots$$

$$
\begin{aligned}
& a\,(x^3 + x^2 + x + 1) \\
+\ & b\,(\quad x^2 \quad + 1) \\
+\ & c\,(x^3 + 2x^2 + x \quad) \\
+\ & d\,(\quad x^2 + 2x + 1) \\
=\ & (\quad -2x + 1)
\end{aligned}
\qquad
\begin{cases}
a + c = 0 \\
a + b + 2c + d = 0 \\
a + c + 2d = -2 \\
a + b + d = 1
\end{cases}
$$

$$\begin{cases} c = -1/2 \\ a = 1/2 \\ d = -1 \\ b = 3/2 \end{cases} \qquad \dots = \int \frac{1}{2}\frac{1}{x+1} + \frac{3}{2}\frac{1}{(x+1)^2} + \frac{-1/2\,x - 1}{x^2+1}$$

$$= \frac{1}{2}\log|x+1| - \frac{3}{2}\frac{1}{x+1} - \frac{1}{4}\log(x^2+1) - \arctan(x) + c$$

7 $$\int_0^1 \frac{1}{x^3+1} = \int_0^1 \frac{1}{(x+1)(x^2-x+1)} = \int_0^1 \frac{a}{x+1} + \frac{cx+d}{x^2-x+1} =$$

$$\begin{aligned} &a(x^2 - x + 1) \\ &+c(x^2 + x) \\ &+d(x+1) \\ &= 1 \end{aligned} \qquad \begin{cases} a + c = 0 \\ -a + c + d = 0 \\ a + d = 1 \end{cases} \qquad \begin{cases} c = -a \\ -2a + d = 0 \\ a + d = 1 \end{cases}$$

$$\begin{cases} a = 1/3 \\ d = 2/3 \\ c = -1/3 \end{cases} \qquad = \int_0^1 \frac{1}{3}\left(\frac{1}{x+1} - \frac{x-2}{(x-\frac{1}{2})^2 + \frac{3}{4}}\right)$$

$$= \frac{1}{3}\log|x+1|\Big|_0^1 - \frac{1}{3}\int_0^1 \frac{\cancel{x}(x-\frac{1}{2}) + \cancel{?}\ {-\tfrac{3}{2}}}{\underbrace{(x-\frac{1}{2})^2 + \frac{3}{4}}_{x^2-x+1}}$$

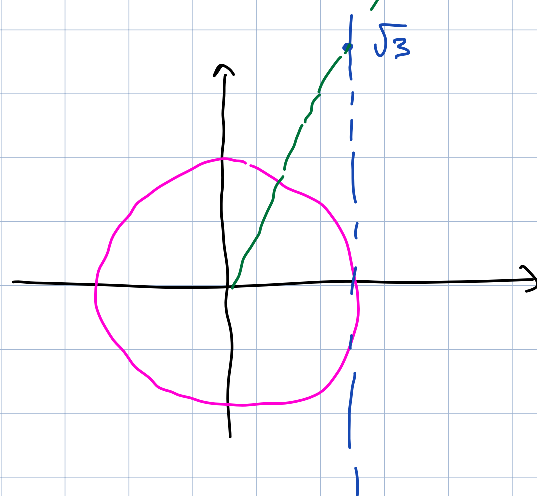
$$= \frac{1}{3}\log(2) + \underbrace{\frac{1}{2}\cdot\frac{1}{2}\log(x^2-x+1)\Big|_0^1}_{0-0} + \frac{1}{2}\int_0^1 \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}$$

$$= \frac{1}{3}\log(2) + \frac{1}{2}\cdot\frac{4}{3}\int_0^1 \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}$$

$$= \frac{1}{3}\cdot\log(2) + \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\Big|_0^1$$

$$= \frac{1}{3}\log(2) + \frac{1}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan\left(-\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{3}\log(2) + \frac{2\pi}{3\sqrt{3}}$$



8 $$\int \frac{1+e^{-x}}{\cosh(x)} = 2\int \frac{1+e^{-x}}{e^{x}+e^{-x}}\,dx$$

$$y = e^{x} \qquad x = \log(y) \qquad dx = \frac{1}{y}\,dy$$

$$= 2\cdot\int \frac{1+\frac{1}{y}}{y+\frac{1}{y}}\cdot\frac{1}{y}\,dy = 2\int \frac{y+1}{y(y^2+1)}\,dy$$

$$= 2\int\left(\frac{a}{y} + \frac{by+c}{y^2+1}\right)dy$$

$a(y^2 \quad +1)$ $\quad a+b=0 \quad b=-2$

$b(y^2)$ $\quad a=2$

$c(\quad y \quad)$ $\quad c=2$

$= 2y+2$

$$= 2\int \frac{1}{y} + \frac{1-y}{y^2+1}$$

$$= 2\log(y) - \int \frac{2y}{y^2+1} + 2\int \frac{1}{y^2+1}$$

$$= 2\log(y) - \log(y^2+1) + 2\arctan(y)$$

$$= 2x - \log(e^{2x}+1) + 2\arctan(e^x) + c$$

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(15) Conviene calcolare per sostituzione?

- $\int (3x+1)^3 dx \qquad y = 3x+1 \qquad x = \frac{1}{3}(y-1) \quad dx = \frac{1}{3}dy$

$$= \int y^3 \cdot \frac{1}{3} dy = \frac{1}{12} y^4 + c = \frac{1}{12}(3x+1)^4 + c$$

$$\frac{1}{4}\cdot\frac{1}{3}(3x+1)^4$$

- $\int (3x^2+1)^3 \, dx$

$y = 3x^2+1 \qquad x = \sqrt{\frac{y-1}{3}}$

$dy = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{y-1}{3}}} \cdot \frac{1}{3} = \ldots$

$= k \cdot \int \frac{y^3}{\sqrt{y-1}} \, dy$

Non conviene

$\int (27x^6 + 27x^4 + 9x^2 + 1) \, dx = \ldots$

- $\int x \cdot (3x^2+1)^3 \, dx = \int (\underbrace{3x^2+1}_{y})^3 \underbrace{x\,dx}_{\frac{1}{6}dy}$

$= \frac{1}{4} \cdot \frac{1}{6} (3x^2+1)^4$

(19) $\int_0^{\pi/4} \cos(2x) \cdot \sin(x) \, dx = \int_0^{\pi/4} (2\cos^2(x) - 1) \sin(x) \, dx$

$= 2 \cdot \frac{1}{3} \cdot (-1) \cos^3(x) + \cos(x) \Big|_0^{\pi/4}$

$= -\frac{1}{6} \cos^3(x) + \cos(x) \Big|_0^{\pi/4} = -\frac{1}{6} \cdot \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{6} + 1\right) = \ldots$

(22) $\int \left(1+\sin^2(x)\right)^3 \cdot \sin(2x)\, dx = \frac{1}{4} \cdot \left(1+\sin^2(x)\right)^4 + c$

(23) $\int_e^{e^2} \frac{1}{x \cdot \log^2(x)}\, dx = -\frac{1}{\log(x)} \Big|_e^{e^2} = -\frac{1}{2}+1 = \frac{1}{2}$

$y = \log(x) \qquad x = e^y \qquad dx = e^y\, dy$

$x = e \longrightarrow y = 1 \qquad x = e^2 \longrightarrow y = 2$

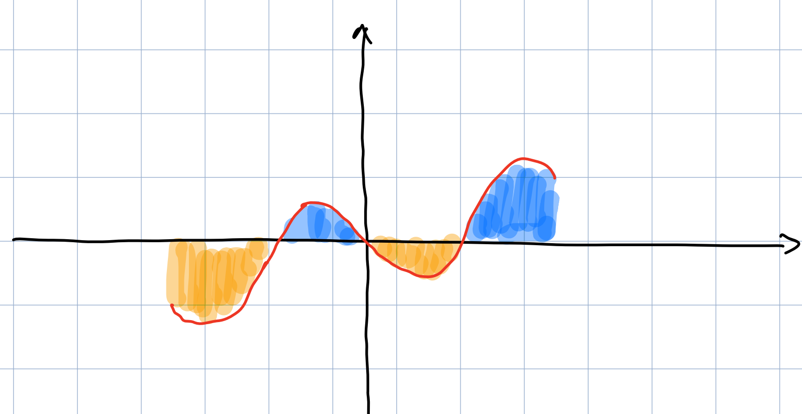
$\int_1^2 \frac{1}{\cancel{e^y} \cdot y^2} \cdot \cancel{e^y}\, dy = -\frac{1}{y}\Big|_1^2 = -\frac{1}{2}+1 = \frac{1}{2}$

(24) $\int \tan^2(x)\, dx = \boxed{\int \left(1+\tan^2(x) - 1\right) dx}$

$= \tan(x) - x + c$

(25) $\int_{-1}^{1} \frac{x}{1+x^2}\, dx = \frac{1}{2} \log(1+x^2) \Big|_{-1}^{1} = \frac{1}{2}\log(2) - \frac{1}{2}\log(2) = 0$

dispari

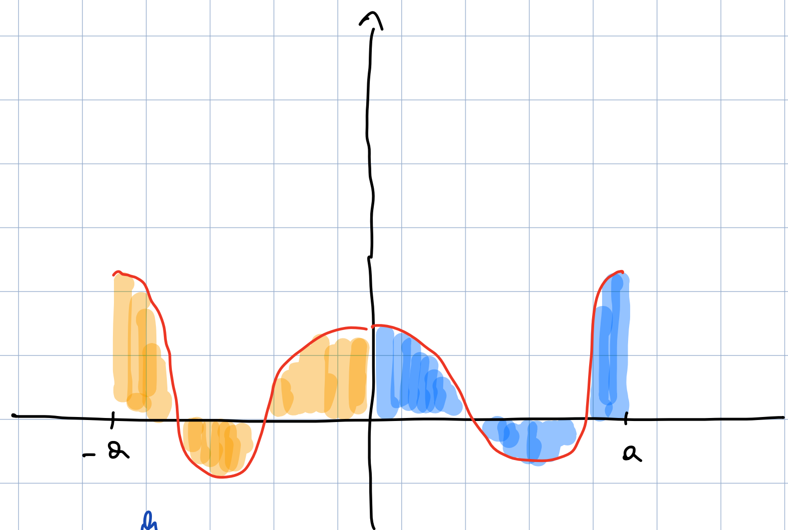


$\int_{-a}^{a} \text{dispari} = 0$

(26) $$\int_{-1}^{1} e^{-|x|}\,dx = \int_{-1}^{0} e^{x}\,dx + \int_{0}^{+1} e^{-x}\,dx$$

$$= e^{x}\Big|_{-1}^{0} - e^{-x}\Big|_{0}^{1} = 1-\frac{1}{e}-\left(\frac{1}{e}-1\right) = 2\left(1-\frac{1}{e}\right)$$

pari



$$\int_{-a}^{a} \text{pari} = 2\cdot\int_{0}^{a}(\text{lei})$$

(30) $$\int \frac{dx}{x\cdot \log(x)} = \log(\log(x)) + c$$

$y = \log(x) \qquad x = e^{y} \qquad dx = e^{y}\,dy$

$$\int \frac{e^{y}\,dy}{e^{y}\cdot y} = \int \frac{1}{y}\,dy = \log(y) + c = \log(\log(x)) + c$$

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$$\int_1^3 x^2\cdot\log(x)\,dx = \frac{1}{3}x^3\cdot\log(x)\Big|_1^3 - \int_1^3 \frac{1}{3}x^3\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{3}x^3\cdot\log(x) - \frac{1}{9}x^3\Big|_1^3 = 9\log(3) - 3 - 0 + \frac{1}{9}$$

$$= 9\log(3) - \frac{26}{9}$$

$$y = \log(x) \qquad x = e^y \qquad dx = e^y\,dy$$

$$x = 1 \longrightarrow y = 0 \qquad x = 3 \longrightarrow \log(3)$$

$$\int_0^{\log(3)} e^{2y}\cdot y\cdot e^y\,dy = \int_0^{\log(3)} y\cdot e^{3y}\,dy$$

$$= \frac{1}{3}e^{3y}\cdot y\Big|_0^{\log(3)} - \int_0^{\log(3)} \frac{1}{3}e^{3y}\,dy = \dots$$

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$$\int e^x\cdot\sin^2(x)\,dx = \frac{1}{2}\int e^x\cdot(1-\cos(2x))\,dx$$

$$= \frac{1}{2}e^x - \frac{1}{2}\underbrace{\boxed{\int e^x\cdot\cos(2x)\,dx}}_{e^x\cdot\cos(2x) + 2\int e^x\cdot\sin(2x)\,dx}$$

$$\boxed{e^x\cdot\cos(2x) + 2e^x\cdot\sin(2x)} - \boxed{2\cdot 2\cdot\int e^x\cdot\cos(2x)\,dx}$$

$$\int e^x\cdot\cos(2x) = \frac{1}{5}\left(e^x\cdot\cos(2x) + 2e^x\cdot\sin(2x)\right)$$