Ist. Mat. I – CIA
11/5/23

Foglio 10.

(1) Trovare base di $W^\perp$.

(A) $W = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 3 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 5 \\ -5 \\ -5 \end{pmatrix}}, \begin{pmatrix} 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} \right\rangle \subset \mathbb{R}^4$

$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

$\dim(W^\perp) = 4 - \dim(W)$

$2 \cdot I - 5 \cdot II$

$= \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle .$

I metodo: $W^\perp = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : \begin{array}{l} x - 2y + z + 3t = 0 \\ -y + z + t = 0 \end{array} \right\}$

$\begin{cases} y = z + t \\ x = 2z + 2t - z - 3t \end{cases}$ $\quad \begin{cases} x = z - t \\ y = z + t \end{cases}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

II metodo: $\left\langle \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right\rangle^\perp$

Cerco $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ ortog. ai due dati.

Nota: $\begin{pmatrix}0\\y\\z\\w\end{pmatrix}$ ortog. ai due dati $\iff \begin{pmatrix}y\\z\\w\end{pmatrix} \perp \begin{pmatrix}b\\c\\d\end{pmatrix}, \begin{pmatrix}f\\g\\h\end{pmatrix}$

quindi posso prendere $\begin{pmatrix}0\\ \begin{pmatrix}b\\c\\d\end{pmatrix}\times\begin{pmatrix}f\\g\\h\end{pmatrix}\end{pmatrix}$

Analog. $\begin{pmatrix}x\\y\\z\\0\end{pmatrix} \perp \begin{pmatrix}a\\b\\c\\d\end{pmatrix}\begin{pmatrix}e\\f\\g\\h\end{pmatrix} \iff \begin{pmatrix}x\\y\\z\end{pmatrix} \perp \begin{pmatrix}a\\b\\c\end{pmatrix}\begin{pmatrix}e\\f\\g\end{pmatrix}$

quindi posso prendere $\begin{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}\times\begin{pmatrix}e\\f\\g\end{pmatrix}\\ 0\end{pmatrix}$.

(B) $W = \left\{ \begin{pmatrix}x\\y\\z\\t\end{pmatrix} \in \mathbb{R}^4 : 2x-y+t=0 \ , \ z-t=0 \right\}$

$\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=0 \qquad \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=0$

$\Longrightarrow W^\perp = \left\langle \begin{pmatrix}2\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix} \right\rangle$

(C) $W = \operatorname{Ker}\begin{pmatrix}1 & -2 & 1\\ 2 & -1 & -1\\ -1 & 5 & -4\end{pmatrix}$

$\det = 4-2+10-1-16+5=0$

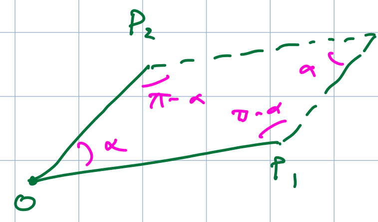
$W = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{array}{l} x-2y+z=0\\ 2x-y-z=0\end{array} \right\}$

$W^\perp = \left\langle \begin{pmatrix}1\\-2\\1\end{pmatrix} \begin{pmatrix}2\\-1\\-1\end{pmatrix} \right\rangle$

$\left(W = \left\langle \begin{pmatrix}1\\-2\\1\end{pmatrix} \wedge \begin{pmatrix}2\\-1\\-1\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}3\\3\\3\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \right.$

② Trovare misure lati, aree, angoli del parallelogramma di vertici O, P_1, P_2, P_1+P_2

Ⓐ $P_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ $\quad P_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$



Lati: $\|P_1\| = \sqrt{14}$

$\|P_2\| = \sqrt{10}$

Area $\|P_1 \times P_2\| = \left\| \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -11 \\ -1 \\ 3 \end{pmatrix} \right\| = \sqrt{131}$

Angoli: $\alpha = \dfrac{P_1 \cdot P_2}{\|P_1\| \cdot \|P_2\|} = -\dfrac{3}{\sqrt{140}}$

Fatto: le soluzioni di $ax''+bx'+cx=0$ sono le combinazioni lineari di x_1, x_2 così ottenute:

λ_1, λ_2 radici $a\lambda^2+b\lambda+c=0$

- $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$; $\quad x_1(t) = e^{\lambda_1 t}$, $x_2(t) = e^{\lambda_2 t}$ $\quad (\Delta > 0)$
- $\lambda_1 = \lambda_2 \in \mathbb{R}$; $\quad x_1(t) = e^{\lambda t}$, $x_2(t) = t \cdot e^{\lambda t}$ $\quad (\Delta = 0)$
- $\lambda_{1,2} = \alpha \pm i\beta$, $\beta \neq 0$; $\quad x_1(t) = \cos(\beta t)\, e^{\alpha t}$
  $x_2(t) = \sin(\beta t)\, e^{\alpha t}$

Esempi: (1) $6x''+x'-15x=0$

$$6\lambda^2+\lambda-15=0 \qquad \lambda_{1,2}=\frac{-1\pm\sqrt{1+360}}{12}=\frac{-1\pm 19}{12} \begin{cases} \frac{3}{2} \\ -\frac{5}{3} \end{cases}$$

Soluzioni: $x(t)=c_1\cdot e^{\frac{3}{2}t}+c_2\cdot e^{-\frac{5}{3}t}$; verifica:

$$6\left(c_1\cdot\frac{9}{4}\cdot e^{\frac{3}{2}t}+c_2\cdot\frac{25}{9}e^{-\frac{5}{3}t}\right) + \left(c_1\cdot\frac{3}{2}\cdot e^{\frac{3}{2}t}-c_2\cdot\frac{5}{3}e^{-\frac{5}{2}t}\right) - 15\left(c_1 e^{\frac{3}{2}t}+c_2\cdot e^{-\frac{5}{3}t}\right)$$

$$= c_1\cdot e^{\frac{3}{2}t}\underbrace{\left(\frac{27}{2}+\frac{3}{2}-15\right)}_{0}+c_2\cdot e^{-\frac{5}{3}t}\underbrace{\left(\frac{50}{3}-\frac{5}{3}-15\right)}_{0}$$

(2) $9x''+12x'+4x=0$

$$9\lambda^2+12\lambda+4=0 \qquad \lambda_{1,2}=\frac{-6\pm\sqrt{36-36}}{9}=-\frac{2}{3}$$

Soluzioni sono $x(t)=(c_1+tc_2)e^{-\frac{2}{3}t}$

Infatti:

$$x'(t)=c_2\cdot e^{-\frac{2}{3}t}-\frac{2}{3}(c_1+tc_2)e^{-\frac{2}{3}t} = \left(-\frac{2}{3}c_1+c_2-\frac{2}{3}tc_2\right)e^{-\frac{2}{3}t}$$

$$x''(t)=-\frac{2}{3}c_2\cdot e^{-\frac{2}{3}t}-\frac{2}{3}\left(-\frac{2}{3}c_1+c_2-\frac{2}{3}tc_2\right)e^{-\frac{2}{3}t} = \left(\frac{4}{9}c_1-\frac{4}{3}c_2+\frac{4}{9}tc_2\right)e^{-\frac{2}{3}t}$$

$$9x''(t)+12x'(t)+4x(t)=e^{-\frac{2}{3}t}\cdot\left(\cancel{4c_1}-4c_2+4\cancel{tc_2}-\cancel{8c_1}+12c_2-\cancel{8tc_2}+\cancel{4c_1}+\cancel{4tc_2}\right)$$

(perso qualcosa)

(3) $4x'' - 12x' + 45x = 0$

$$4\lambda^2 - 12\lambda + 45 = 0$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36 - 180}}{4} = \frac{6 \pm \sqrt{-144}}{4} = \frac{6 \pm 12i}{4} = \frac{3}{2} \pm 3i$$

Soluzioni : $x(t) = k \cdot e^{\frac{3}{2}t} \cdot \cos(3t) + h \cdot e^{\frac{3}{2}t} \cdot \sin(3t)$

Infatti:

$$x'(t) = \frac{3}{2} k e^{\frac{3}{2}t} \cos(3t) - 3k e^{\frac{3}{2}t} \sin(3t) + \frac{3}{2} h e^{\frac{3}{2}t} \sin(3t) + 3h e^{\frac{3}{2}} \cdot \cos(3t)$$

$$= e^{\frac{3}{2}t} \cdot \left(\left(\frac{3}{2} k + 3h \right) \cos(3t) + \left(-3k + \frac{3}{2} h \right) \sin(3t) \right)$$

$$x''(t) = e^{\frac{3}{2}t} \cdot \left(\left(\frac{9}{4} k + \frac{9}{2} h - 9k + \frac{9}{2} h \right) \cos(3t) \right.$$
$$\left. \left(-\frac{9}{2} k + \frac{9}{4} h - \frac{9}{2} k - 9h \right) \sin(3t) \right)$$

$$4x'' - 12x' + 45x = .$$

$$= e^{\frac{3}{2}t} \left(\cos(3t) \cdot \left(\cancel{9k} + 18h \cancel{-36k} + 18h \cancel{-18k} - 36h \cancel{+45k} \right) \right.$$
$$\left. \sin(3t) \cdot \left(-18k + 9h - 18k - 36h - 36k + 18h + 45h \right) \right)$$

$$\begin{cases} y' = f(x) \cdot g(y) \\ y(x_0) = y_0 \end{cases}$$

$g(y_0) = 0 \implies$ ho soluz. costante $y = y_0$

$g(y_0) \neq 0$ la soluzione si trova esplicitando y in funzione di x da

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

$$H(y) = F(y) + c$$

c si trova usando $y(x_0) = y_0$

$$H'(y_0) = \frac{1}{g(y_0)} \neq 0 \implies$$ localmente H è invertibile

③ Ⓐ $$y' = \frac{x}{1+\log(y)}$$

non ci sono soluzioni costanti

$$\int (1+\log(y))\,dy = \int x\,dx = \frac{1}{2}x^2 + c$$

$t = \log(y)$ $\qquad y = e^t$ $\qquad dy = e^t\,dt$

$$\int (1+\log(y))\,dy = \int (1+t)\,e^t\,dt = (1+t)\,e^t - \int e^t = t\,e^t$$

$$= y \cdot \log(y)$$

Soluzione: ricavare y come funzione di x da

$$y \cdot \log(y) = \frac{1}{2}x^2 + c$$ (non esplicitabile)

Ⓑ $$y' = x\left(1 + \frac{1}{y}\right)$$

Soluzione costante $y = -1$.

$$\int \frac{y}{y+1}\,dy = \int x\,dx = \frac{1}{2}x^2 + c$$

$$\int \left(1 - \frac{1}{y+1}\right) dy = y - \log|y+1| \qquad \text{(non si esplicita)}$$

Ⓒ $y' = \dfrac{x-3}{\sin(y)}$ No soluz. costanti

$$\int \sin(y)\,dy = \int (x-3)\,dx$$

$$-\cos(y) = \frac{1}{2}x^2 - 3x + c$$

$$y = \arccos\left(-\frac{1}{2}x^2 + 3x + c\right)$$

Ⓓ $y' = e^{x-y}$ No soluzioni costanti

$$\int e^y\,dy = \int e^x\,dx$$

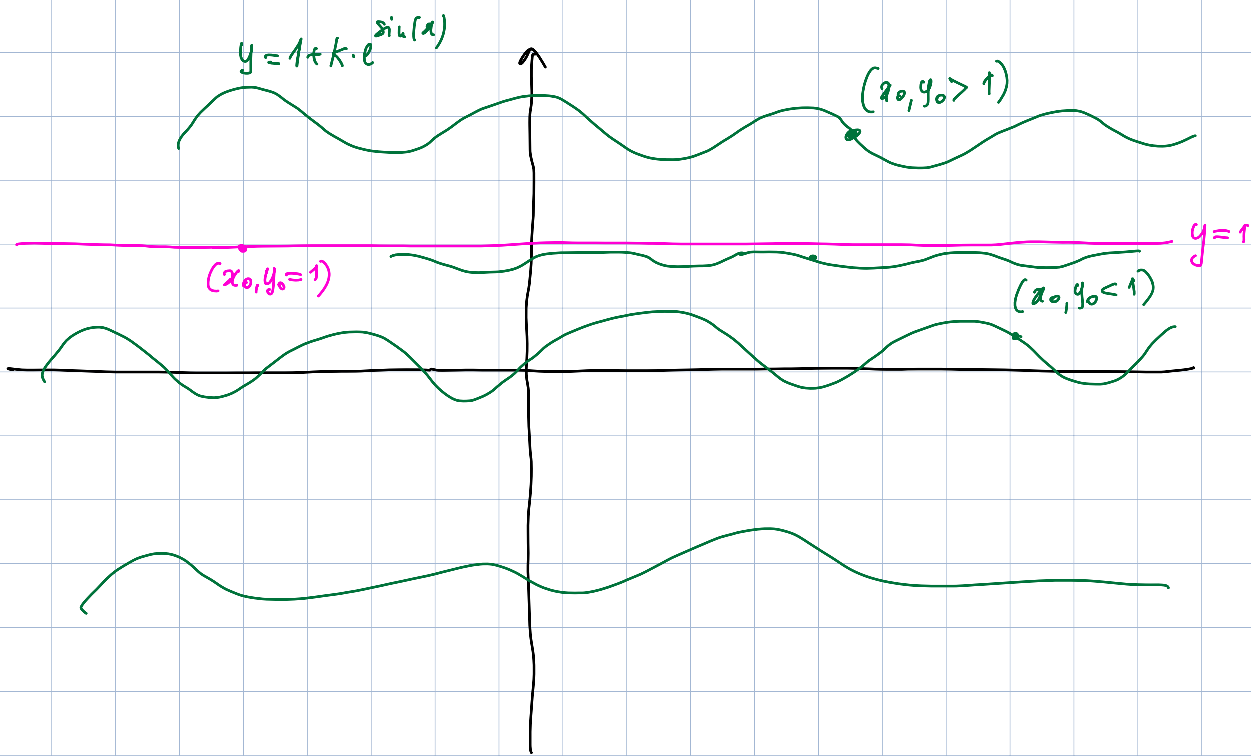
$$e^y = e^x + c \qquad y = \log(e^x + c)$$

Ⓔ $y' = (y-1)\cos(x)$ Soluz. costante $y=1$

$$\int \frac{dy}{y-1} = \int \cos(x)\,dx$$

$$\log|y-1| = \sin(x) + c$$

$$|y-1| = e^{\sin(x)+c} = k\cdot e^{\sin(x)}$$



Equazioni lineari non omogenee del I ordine:

$$y' = a(x)\cdot y + b(x)$$

- si risolve $y' = a(x)\cdot y$; $\int \frac{dy}{y} = \int a(x)\,dx$

$$\log|y| = A(x) + c \qquad |y| = k\,e^{A(x)}$$

$$y = k\cdot e^{A(x)}$$

- Si risolve quella non omogenea con variazione delle costanti:

cerco $y(x) = k(x) \cdot e^{A(x)}$. Funziona:

Voglio: $y' = a \cdot y + b$

cioè $k' \cdot e^{A} + \cancel{k \cdot e^{A} \cdot a} = \cancel{a \cdot k \cdot e^{A}} + b$

cioè $k' = e^{-A} \cdot b \Rightarrow k = \int e^{-A} \cdot b$

4 A $y' = -2xy + x$

$y' = x(1-2y)$ $\quad \int \frac{1}{1-2y} = \int x$

$-\cancel{\frac{1}{2}} \log |1-2y| = \cancel{\frac{1}{2}} x^2 + c$

$|1-2y| = k e^{-x^2}$

$$\boxed{y = \frac{1}{2} + k e^{-x^2}}$$

$y' = -2xy$ $\quad y(x) = k \cdot e^{-x^2}$

$k' \cdot e^{-x^2} + \cancel{k(-2x e^{-x^2})} = \cancel{-2x \cdot k e^{-x^2}} + x$

$k' = x \cdot e^{x^2}$ $\quad k = \frac{1}{2} e^{x^2} + c$

$$\Rightarrow \boxed{y = \frac{1}{2} + c \cdot e^{-x^2}}$$

$\boxed{B}$ $\quad y' = \dfrac{y}{1+x^2}$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x^2}$$

$$\log|y| = \arctan(x) + c$$

$$y = k \cdot e^{\arctan(x)}$$

$\boxed{C}$ $\quad y' = \dfrac{y}{1-x^2} + (1-x)$

$$y' = \frac{y}{1-x^2}$$

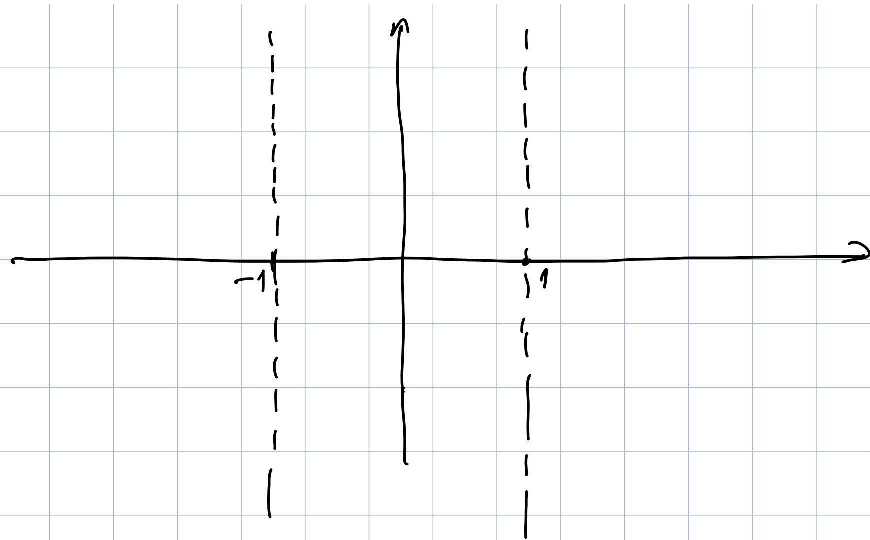
$$\int \frac{dy}{y} = \int \frac{dx}{1-x^2} = \frac{1}{2}\int \left(\frac{1}{1-x} + \frac{1}{1+x} \right) dx$$

$$\int \frac{dy}{y} = \log|y|$$

$$\frac{1}{2}\int \left(\frac{1}{1-x} + \frac{1}{1+x} \right) dx = \frac{1}{2}\left(\log|1+x| - \log|1-x| \right) + c = \log\left|\frac{1+x}{1-x}\right|^{1/2} + c$$

$$y = k \cdot \left|\frac{1+x}{1-x}\right|^{1/2}$$

(prendiamo $-1 < x < 1$)



Risolviamo $y' = \frac{y}{1-x^2} + (1-x)$

$$y(x) = k(x)\cdot\sqrt{\frac{1+x}{1-x}}$$

$$k'(x)\cdot\sqrt{\frac{1+x}{1-x}} + k(x)\cdot\cancel{\frac{1}{2}}\cdot\frac{1}{\sqrt{\frac{1+x}{1-x}}}\cdot\frac{\cancel{1-x+1+x}}{(1-x)^2} =$$

$$= \frac{1}{\underbrace{1-x^2}_{(1-x)(1+x)}}\cdot k(x)\cdot\sqrt{\frac{1+x}{1-x}} + (1-x)$$

$$k'(x) = \sqrt{\frac{1-x}{1+x}}\cdot(1-x) = \frac{(1-x)^{3/2}}{(1+x)^{1/2}}$$

$$k(x) = \dots + c$$

$\boxed{D}$ $$y' = -\frac{y}{x} - \frac{e^{-x}}{x}$$

Trovare tutte le soluzioni $y: I \to \mathbb{R}$ dell'equazione data specificando I.

<u>omog.</u> $$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\log|y| = -\log|x|$$

$$|y| = k \cdot \frac{1}{|x|} \qquad y = \frac{k}{x}$$

<u>non omog</u> $$y(x) = \frac{k(x)}{x}$$

$$\frac{k'(x)}{x} - \cancel{\frac{k(x)}{x^2}} = -\cancel{\frac{k(x)}{x} \cdot \frac{1}{x}} - \frac{e^{-x}}{x}$$

$$k'(x) = -e^{-x} \qquad k(x) = e^{-x} + c$$

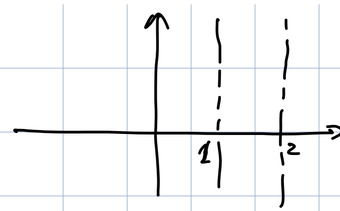
$$y(x) = \frac{e^{-x} + c}{x}$$

per ogni $(x_0, y_0) \in \mathbb{R}^2$ con $x_0 \neq 0$ esiste un'unica soluzione dell'equazione t.c. $y(x_0) = y_0$, $y: I \to \mathbb{R}$ con

$$I = (-\infty, 0) \quad \text{se } x_0 < 0$$
$$I = (0, +\infty) \quad \text{se } x_0 > 0$$

$$c = x_0 y_0 - e^{-x_0}$$

(E) $y' = \frac{x}{x^2-3x+2}\cdot y - \sin(x)$



Omog

$$\int \frac{dy}{y} = \int \frac{x}{x^2-3x+2}\,dx = \int \frac{2}{x-2} - \frac{1}{x-1}$$

$$\log|y| = 2\log|x-2| - \log|x-1| \qquad \begin{cases} a+b=1 \\ -a-2b=0 \end{cases} \qquad \begin{matrix} a=2 \\ b=-1 \end{matrix}$$

$$y = k\cdot\frac{(x-2)^2}{x-1}$$

non omog: $\quad y(x) = k(x)\cdot\frac{(x-2)^2}{x-1}$

$$k'(x)\cdot\frac{(x-2)^2}{x-1} + k(x)\cdot\frac{\overbrace{2(x-2)(x-1)-(x-2)^2}^{(x-2)(2x-2-x+2)}}{(x-1)^2} =$$

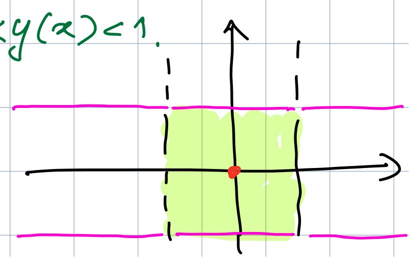
$$= \frac{x}{(x-1)(x-2)}\cdot k(x)\cdot\frac{(x-2)^2}{(x-1)} - \sin(x)$$

$$k'(x) = -\frac{(x-1)\cdot\sin(x)}{(x-2)^2} \qquad \ldots\ldots$$

$\boxed{5}$ (A) $\begin{cases} y' = \frac{y^2-1}{x^2-1} \\ y(0)=0 \end{cases}$ Soluzioni costanti $y=\pm 1$ non soddisfano $y(0)=0$.

Oss: le soluzione max $y: I \to \mathbb{R}$, $I \subset (-1,1)$ e $-1<y(x)<1$.

$$\int \frac{1}{1-y^2}\,dy = \int \frac{1}{1-x^2}\,dx$$



$$\frac{1}{2}\int\left(\frac{1}{1-y}+\frac{1}{1+y}\right)dy = \frac{1}{2}\int\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx$$

$$\log\frac{1+y}{1-y} = \log\frac{1+x}{1-x}+c$$

$$\frac{1+y}{1-y} = k\cdot\frac{1+x}{1-x}$$

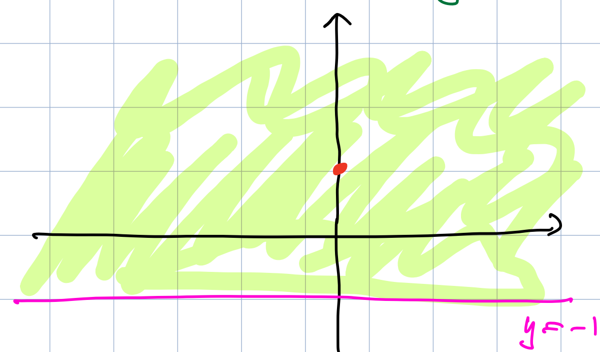
condiz. iniz. $\frac{1+0}{1-0} = k\cdot\frac{1+0}{1-0}$

$\Rightarrow k = 1$

$\Rightarrow y(x) = x.$

B $\begin{cases} y' = \sqrt{(1+y)(1+x^2)} \\ y(0) = 1 \end{cases}$

Soluz. cost. $y=-1$



$$\int\frac{dy}{\sqrt{1+y}} = \int\sqrt{1+x^2}\,dx$$

$$\parallel$$

$$\frac{1}{2}\sqrt{1+y}$$

$$\int\sqrt{1+x^2}\,dx \qquad \begin{aligned} x &= \sinh(t) \\ dx &= \cosh(t) \end{aligned}$$

$$= \boxed{\int\cosh^2(t)\,dt} = \cosh(t)\cdot\sinh(t) - \int\sinh^2(t)\,dt$$

$$= \cosh(t)\cdot\sinh(t) - \boxed{\int\cosh^2(t)\,dt} + t$$

$$= \frac{1}{2}\left(t + \cosh(t)\cdot\sinh(t)\right) + c$$

$$t = \operatorname{arcsinh}(x) = \log\left(x + \sqrt{1+x^2}\right)$$

$$\cancel{\frac{1}{2}}\sqrt{1+y} = \cancel{\frac{1}{2}}\left(\boxed{\log\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2}}\right) + c$$

$$\sqrt{2} = 0 + 0 + c \qquad c = \sqrt{2}$$

$$y = \left(\boxed{} + \sqrt{2}\right)^2 - 1$$

C
$$\begin{cases} y' = \dfrac{y}{x+2} + \dfrac{1}{4x} \\ y(-1) = 1 \end{cases}$$

escludo passaggio per $x=0$ e $x-2$

$y: I \to \mathbb{R} \quad I \subset (-2, 0)$

omog. $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x+2}$

$\log(y) = \log(x+2) + c$

$y = k\cdot(x+2)$

non omog. $\quad y = k(x)\cdot(x+2)$

$$k'(x)\cdot(x+2) + \cancel{k(x)} = \cancel{\frac{k(x)\cdot(x+2)}{x+2}} + \frac{1}{4x}$$

$$k'(x) = \frac{1}{4x(x+2)} = \frac{1}{4}\cdot\frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)$$

$$k(x) = \frac{1}{8}\cdot\log\frac{-x}{x+2} + c$$

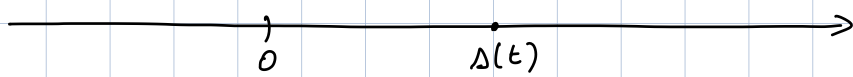
$$y(x) = \frac{x+2}{8}\left(\log\left(-\frac{x}{x+2}\right)+c\right)$$

$$1 = \frac{1}{8}\cdot\left(\log(1)+c\right)$$

$$y(x) = \frac{x+2}{8}\left(\log\left(-\frac{x}{x+2}\right)+8\right)$$

definita su $(-2,0)$

$\boxed{6}$



$$v(t) = s'(t)$$

Trovare posizione al tempo $t = 5$ se

$$\begin{cases} v(t) = -\log(t)\cdot s(t) + \log(t) \\ s(1) = 2 \end{cases}$$

$$\begin{cases} s'(t) = \log(t)\left(1-s(t)\right) \\ s(1) = 2 \end{cases}$$

$$\int \frac{ds}{1-s} = \int \log(t)\,dt$$

$$-\log(s-1) = t\cdot\log(t) - t + c$$

$$\log(2-1) = 1\cdot\log(1) - 1 + c \Rightarrow c = 1$$

$$s-1 = e^{t(\log(t)-1)+1}$$

$$s = 1 + e^{t(\log(t)-1)+1} \qquad t = 5 \ldots$$