Ist. Mat. I - CLA
16/3/23

Regola di Cramer

$A \cdot x = b \qquad A \in \mathbb{R}^{n\times n} \qquad$ con $\quad \det(A) \neq 0$

la soluzione x è :

$$x_i = \frac{\det(A_i)}{\det(A)}$$

A_i = matrice ottenuta da A sostituendo con b la i-esima colonna

Es:

$$\begin{cases} 7x - 5y = 4 \\ 3x + 2y = -1 \end{cases}$$

$$x = \frac{\begin{vmatrix} 4 & -5 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 7 & -5 \\ 3 & 2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 7 & 4 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 7 & -5 \\ 3 & 2 \end{vmatrix}}$$

Es:

$$\begin{cases} 7x + 4y - 3z = 2 \\ -4x + y + 2z = 5 \\ 9x - 6y + 17z = -4 \end{cases}$$

$$z = \frac{\begin{vmatrix} 7 & 4 & 2 \\ -4 & 1 & 5 \\ 9 & -6 & -4 \end{vmatrix}}{\begin{vmatrix} 7 & 4 & -3 \\ -4 & 1 & 2 \\ 9 & -6 & 17 \end{vmatrix}}$$

Spiegazione: $x = A^{-1} \cdot b$

$$x_i = (A^{-1} \cdot b)_i = \sum_{j=1}^{n} (A^{-1})_{ij} \cdot b_j$$

$$= \frac{1}{\det(A)} \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ji}) \cdot b_j$$

$$= \frac{1}{\det(A)} \cdot \boxed{\underbrace{\sum_{j=1}^{n} (-1)^{j+i} \cdot \underset{\text{se fosse } a_{ji}}{b_j} \cdot \det(A_{ji})}_{\det(A) \text{ calcolato con sviluppo lungo colonna } i}}$$

$$= \det(A_i)$$

$$A_i = \begin{array}{c} \begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix} \\ \qquad\qquad\uparrow i \end{array}$$

Foglio 6.

4c

$$\begin{cases} 2x - 2y + z + 4w = 0 \\ x - y - 4z + 2w = 0 \\ -x + y + 3z - 2w = 0 \\ 3x - 3y + z + 6w = 0 \end{cases}$$

i-esima equaz $\rightsquigarrow$ $k \cdot$ (i-esima) $+ h \cdot$ (j-esima) $\quad k \neq 0, \; j \neq i$

sistema equivalente

$$\begin{matrix} I+III \\ I/2 \\ II \\ IV/3 \end{matrix}\begin{cases} z=0 \\ x-y+2w=0 \\ x-y+2w=0 \\ x-y+2w=0 \end{cases} \qquad \begin{cases} z=0 \\ x-y+2w=0 \end{cases} \qquad \begin{cases} z=0 \\ y=x+2w \end{cases}$$

Soluzioni $\left\{ \begin{pmatrix} x \\ x+2w \\ 0 \\ w \end{pmatrix} : x,w \in \mathbb{R} \right\}$

$$= \left\{ x \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} : x,w \in \mathbb{R} \right\}$$

$$= \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

5b $\begin{cases} x+y+z=k \\ x-ky-z=1 \\ 2x+y+kz=k+1 \end{cases}$

I $\begin{cases} z=k-x-y \\ x-ky-k+x+y=1 \\ 2x+y+k^2-kx-ky=k+1 \end{cases}$

$$\begin{cases} z=k-x-y \\ 2x+(1-k)y=1+k \\ (2-k)x+(1-k)y=-k^2+k+1 \end{cases}$$

$$\begin{matrix} \\ II-III \\ II \end{matrix}\begin{cases} z=k-x-y \\ kx=k^2 \\ (1-k)y=1+k-2x \end{cases}$$

$k=0 \quad \begin{cases} \forall x \\ y = 1-2x \\ z = -x-1+2x = x-1 \end{cases} \qquad \begin{pmatrix} x \\ 1-2x \\ x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\rangle$

$k \neq 0 \quad \begin{cases} x = k \\ z = -y \\ (1-k)y = 1-k \end{cases}$

$k=1 \quad \begin{cases} x=1 \\ \forall y \\ z=-y \end{cases} \qquad \begin{pmatrix} 1 \\ y \\ -y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$k \neq 1 \quad \begin{cases} y = 1 \\ x = k \\ z = -1 \end{cases} \qquad \begin{pmatrix} k \\ 1 \\ -1 \end{pmatrix}$

5b $\quad \begin{cases} x + y + z = k \\ x - ky - z = 1 \\ 2x + y + kz = k+1 \end{cases}$

I $\quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & -k & -1 \\ 2 & 1 & k \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -k-1 & -2 \\ 2 & -1 & k-2 \end{vmatrix} = -\begin{vmatrix} k+1 & -2 \\ 1 & k-2 \end{vmatrix}$

$= -\left(k^2 - k - 2 + 2\right) = k - k^2 = k(1-k)$

$k \neq 0, 1$ soluz unica

$x = \dfrac{\begin{vmatrix} k & 1 & 1 \\ 1 & -k & -1 \\ k+1 & 1 & k \end{vmatrix}}{k(1-k)} = \dfrac{\begin{vmatrix} k+1 & 1-k & 0 \\ 1 & -k & -1 \\ 2k+1 & 1-k^2 & 0 \end{vmatrix}}{k(1-k)} = \dfrac{\begin{vmatrix} k+1 & 1-k \\ 2k+1 & 1-k^2 \end{vmatrix}}{k(1-k)}$

$= \dfrac{(1-k) \cdot \begin{vmatrix} k+1 & 1 \\ 2k+1 & 1+k \end{vmatrix}}{k(1-k)} = \dfrac{k^2 + 2k + 1 - 2k - 1}{k} = k.$

$y = \dots \qquad z = \dots$

$k=0 \quad \begin{cases} x+y+z=0 \\ x-z=1 \\ 2x+y=1 \end{cases} \quad \begin{cases} z = x-1 \\ y = 1-2x \\ x+(1-2x)+(x-1)=0 \end{cases} \quad \begin{cases} y = 1-2x \\ z = x-1 \end{cases}$

$k=1 \quad \dots$

5d
$$\begin{cases} kx - 2(k+1)y + z = 4-2k \\ (k+1)y + z = k+3 \\ 2kx - 5(k+1)y + 2z = 8-3k \end{cases}$$

- Osservo che equivale a

$$\begin{cases} u = kx \\ w = (k+1)y \\ u - 2w + z = 4-2k \\ w + z = k+3 \\ 2u - 5w + 2z = 8-3k \end{cases}$$

Risolvo questo e poi
- $k \neq 0, -1$
  $x = u/k, \quad y = w/(k+1)$
- $k=0 \quad y = w$
  se $u=0 \quad \forall x$
  se $u \neq 0$ impossibile
- $k=-1 \quad x = -u$
  se $w=0 \quad \forall y$
  se $w \neq 0$ impossibile

II $\quad z = -w+k+3$

I $\quad u = 2w - z + 4 - 2k$
$= 2w + w - k - 3 + 4 - 2k$
$= 3w - 3k + 1$

III $\quad 6w - 6k + \cancel{2} - 5w - 2w + 2k + \cancel{6} = \cancel{8} - 3k$

$w = 5k$

$$\begin{cases} u = 12k+1 \\ w = 5k \\ z = 3-4k \end{cases}$$

$k \neq 0, 1$ ...

$k = 0$ $u \neq 0$ impossibile

$k = -1$ $w \neq 0$ impossibile

(5d)

$$\begin{cases} kx - 2(k+1)y + z = 4-2k \\ (k+1)y + z = k+3 \\ 2kx - 5(k+1)y + 2z = 8-9k \end{cases}$$

$$\begin{vmatrix} k & -2(k+1) & 1 \\ 0 & k+1 & 1 \\ 2k & -5(k+1) & 2 \end{vmatrix} = k(k+1) \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 2 & -5 & 2 \end{vmatrix}$$

$$= k(k+1) \begin{vmatrix} 1 & -3 & 1 \\ 0 & 0 & 1 \\ 2 & -7 & 2 \end{vmatrix} = k(k+1)$$

$k \neq 0, -1$ $\rightsquigarrow$ soluz. unica $x = \dfrac{\begin{vmatrix} \vdots & \vdots & \vdots \end{vmatrix}}{k(k+1)}$ ...

$k = 0$ $\begin{cases} -2y + z = 4 \\ y + z = 3 \\ -5y + 2z = 8 \end{cases}$ $\begin{cases} y = 3 - z \\ 3z = 10 \\ 7z = 23 \end{cases}$ impossibile

$k = 1$ ...

Foglio 7

Fatti generali:

- v (uno solo) lin. indip. $\Leftrightarrow$ $v \neq 0$
- v_1, v_2 (due) lin. indip. $\Leftrightarrow$ non sono proporzionali
- $v_1, \dots, v_k$ lin. dip. $\Rightarrow$ $v_1, \dots, v_k, w$ lin. dip.

① Dati $v_1, \dots, v_4$ trovare tutti i $v_{i_1}, \dots, v_{i_k}$ lin. indip.

(a) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- un solo vettore: tutti ok
- due vettori: tutti tranne $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- tre vettori: $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ * no

  inoltre $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ interscambiabili: questa

$\begin{pmatrix} -3 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Ci chiediamo se

$$\begin{cases} -3a - b + c = 0 \\ a + b = 0 \end{cases}$$

ha solo la soluz $a = b = c = 0$.

$$\begin{cases} b = -a \\ c = 2a \end{cases}$$

No: ad es. $a=1$, $b=-1$, $c=2$. Lin. dip.

• quattro vettori: no

(b) $\begin{pmatrix}2\\-1\\0\end{pmatrix} \begin{pmatrix}2\\1\\0\end{pmatrix} \begin{pmatrix}0\\2\\0\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix}$

• uno solo: ok
• due: ok
• tre:

$\begin{pmatrix}2\\-1\\0\end{pmatrix} \quad \begin{pmatrix}2\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\2\\0\end{pmatrix}$

Mi chiedo se $\begin{cases} a+b=0 \\ -a+b+2c=0 \end{cases}$ ha solo soluz. 0:

$\begin{cases} b=-a \\ c=a \end{cases}$ No: $1, -1, 1$. Lin. dip.

Oss: ho scoperto che $v_1 - v_2 + v_3 = 0$

$(v_1 = v_2 - v_3)$

$v_2 = v_1 + v_3$

$v_3 = -v_1 + v_2$

$\Rightarrow$ ogni comb. lin. di v_1, v_3, v_4

o v_2, v_3, v_4

si riscrive come comb. lin. di v_1, v_2, v_4 e il coeff. di v_4 non cambia

$\Rightarrow$ Basterà vedere che v_1, v_2, v_4 sono lin. indip. per concludere che anche $v_1\, v_3\, v_4$, $v_2\, v_3\, v_4$ lo sono.

$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Mi chiedo se: $\begin{cases} 2a + 2b = 0 \\ -a + b + c = 0 \\ c = 0 \end{cases}$

ha solo la soluz. $a = b = c = 0$.

$\begin{cases} c = 0 \\ a + b = 0 \\ -a + b = 0 \end{cases}$ sì. $\underline{\underline{\text{Lin. indip.}}}$

(d) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Verifico che tutti e 4 sono lin. indip. cioè che

$\begin{cases} a + b - 5d = 0 \\ 2c = 0 \\ d = 0 \\ -a + b + c + d = 0 \end{cases}$ ha solo soluz. 0.

$\begin{cases} c = 0 \\ d = 0 \\ a + b = 0 \\ -a + b = 0 \end{cases}$ Sì. $\underline{\underline{\text{Lin. indip.}}}$

(2) Per quali t: $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ t \end{pmatrix}$ sono lin. indip.?

Se $\begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 2 & 3 & t \end{vmatrix} \neq 0$ il sistema $\begin{cases} a + 2b + 3c = 0 \\ \rule{3cm}{0.4pt} \\ \rule{3cm}{0.4pt} \end{cases}$

ha soluz. unica $\Rightarrow a = b = c = 0 \Rightarrow$ lin. indip.

$$\begin{vmatrix} 1 & 3 & 3 \\ 1 & 0 & 0 \\ 2 & 5 & t \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 \\ 5 & t \end{vmatrix} = -3(t-5)$$

$t \neq 5$ lin. dip.

$t = 5 \quad \begin{cases} a + 2b + 3c = 0 \\ a - b = 0 \\ 2a + 3b + 5c = 0 \end{cases} \qquad \begin{cases} b = a \\ 3a + 3c = 0 \\ 5a + 5c = 0 \end{cases}$

$\begin{cases} b = a \\ c = -a \end{cases}$. Soluz. $1, 1, -1 \implies$ lin. dip.

(3) $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ -2 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 1 \end{pmatrix}$

E' vero che $v = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix}$ è comb. lin. di v_1, v_2, v_3, v_4?

$\left(v \overset{?}{\underset{\notin}{\in}} \langle v_1, v_2, v_3, v_4 \rangle \right)$

Se sì, i coeff. sono unici?

$\begin{cases} a + c + 3d = 2 \\ 2a + 3b - c + 2d = -1 \\ -a - b - d = -1 \\ -2b + c + d = 2 \end{cases} \qquad \begin{array}{l} \text{III} \\ \text{IV} \\ \text{I} \\ \text{II} \end{array} \begin{cases} a = 1 - b - d \\ c = 2 + 2b - d \\ 1 - b - d + 2 + 2b - d + 3d = 2 \\ 2 - 2b - 2d + 3b - 2 - 2b + d + 2d = -1 \end{cases}$

$\begin{cases} a = \dots \\ c = \dots \\ b + d = -1 \\ -b + d = -1 \end{cases} \qquad \begin{cases} b = 0 \\ d = -1 \\ a = 2 \\ c = 3 \end{cases}$

$\implies v = 2v_1 + 3v_3 - v_4$

sì appartiene
sì i coeff sono unici

(4) $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

Lin. indip. Per quali t $\begin{pmatrix} 1 \\ -1 \\ 2t-8 \\ t+1 \end{pmatrix} \in \langle v_1, v_2, v_3 \rangle$

Lin. indip. $\begin{cases} a + 2b = 0 \\ -a + b + c = 0 \\ b - c = 0 \\ a = 0 \end{cases}$ $\begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ \dots \end{cases}$ sì

$\begin{cases} a+2b = 1 \\ -a+b+c = -1 \\ b-c = 2t-8 \\ a = t+1 \end{cases}$ $\begin{array}{l} \text{IV} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{cases} a = t+1 \\ b = -t/2 \\ -t-1-t/2+c = -1 \\ -t/2 - c = 2t-8 \end{cases}$

$\text{III} \quad c = \frac{3}{2}t$

$\text{IV} \quad c = -\frac{5}{2}t + 8$

Esiste soluz. se $\frac{3}{2}t = -\frac{5}{2}t + 8$ $t = -2$

$a = -1$ $b = 1$ $c = -3$

(5) $V = \mathbb{R}^{2,2}$ sono sottosp?

(a) $\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$

$k \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} + h \cdot \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} \boxed{} & kb+h\beta \\ kb+h\beta & \boxed{} \end{pmatrix}$

Sì

(b) $\{A : \det(A) = 0\} = W$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$ad - bc = 0$

- $0 \in W$ vero
- $A \in W \Rightarrow k \cdot A \in W$ vero
- $A_1, A_2 \in W \Longrightarrow A_1 + A_2 \in W$

falso:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\det = 0} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\det = 0} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\det = 1}$$

Le equaz. che vanno bene per definire sottospazi sono quelle lineari omogenee in...

- le coord. se in $\mathbb{R}^n$
- i coeff. delle matr. in $\mathbb{R}^{m,n}$
- ...
- "mascherate"

Es: $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : e^{2x-3y} = 1\right\}$

$= \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : 2x - 3y = 0\right\}$

è sottosp.

Es: $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : (3x + 5y)^{19} = 0\right\}$