Knot theory 16/5/2019

Titolo nota 16/05/2019

Tait Conjecture (1st Tait conjecture): let D be a reduced connected (non-split) alternating diagram of a link L. Then D is minimal (it has the least possible number of crossings).

We have already proved that connected, reduced, alternating diagrams are adequate.

From now on, let D be a diagram of a link L.

For any polinomial f, let $M(f)$, $m(f)$ be the maximal (minimal) value appearing as exponent of the variable t in f.

$$M\left(t^{\frac{1}{2}} - t^{\frac{5}{2}}\right) = \frac{5}{2} \qquad m\left(t^{-2} + t\right) = -2.$$

Proposition: ① $m(\langle D\rangle) \geq \dfrac{-n-2|S_+(D)|+2}{4}$, with equality if D is plus-adequate.

② $M(\langle D\rangle) \leq \dfrac{n+2|S_-D|-2}{4}$, with equality if D is minus-adequate. $n = \#$ crossings of D

Warning: In the literature, usually $\langle D\rangle$ is a polynomial in the variable A, s.t.

$A^2 = t^{-\frac{1}{2}}$. Therefore, the statement of this Proposition usually has the right-hand sides multiplied by 4, and the roles of $+/-$ adequacy switched. (Also s_+D, s_-D must be switched)

Proof: We prove ①, the proof of ② being identical.

$$\langle D \rangle = \sum_{s} \langle D | s \rangle = \langle D | s_+ \rangle + \sum_{s \neq s_+} \langle D | s \rangle.$$

$$\langle D | s_+ \rangle = \left(t^{-\frac{1}{4}}\right)^{\overset{\sum_i s_+(i)}{\underset{m}{\shortparallel}}} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^{|s_+(D)|-1}$$

$$m(\langle D | s_+ \rangle) = \left(-\frac{1}{4}\right) \cdot m + \left(-\frac{1}{2}\right)\left(|s_+ D| - 1\right) =$$

$$= \frac{-m - 2|s_+ D| + 2}{4}$$

Now we must show that $m(\langle D | s \rangle) \geq$
$\geq m(\langle D | s_+ \rangle)$ for every (other) states s, and that $m(\langle D | s \rangle) > m(\langle D | s_+ \rangle)$ for $s \neq s_+$, provided that D is plus-adequate.

Let $s \neq s_+$. Then I have a sequence $s_+ = s_0, s_1, s_2, \ldots, s_k = s$ in which s_{i+1} is obtained from s_i by changing the sign of one crossing.

$$\langle D | s \rangle = \left(t^{-\frac{1}{4}}\right)^{\sum_i s(i)} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^{|s(D)|-1}$$

In passing from s_i to s_{i+1}, one crossings passes from $+1$ to -1, hence $\Sigma s(i)$ decreases by 2. Moreover, $|s_{i+1} D| = |s_i D| \pm 1$. This means that

$$m(\langle D|s_{i+1}\rangle) - m(\langle D|s_i\rangle) = \frac{1}{2} \pm \frac{1}{2} = \begin{cases} +1 \\ 0 \end{cases}$$

In any case, $m(\langle D|s_{i+1}\rangle) \geq m(\langle D|s_i\rangle)$ hence by induction $m(\langle D|s\rangle) \geq m(\langle D|s_+\rangle)$ for every $s \Longrightarrow m(\langle D\rangle) \geq m(\langle D|s_+\rangle)$, as desired.

Now, if D is assumed to be adequate, in passing from $s_+ = s_0$ to s_1 we have $|s_+ D| > |s_1 D|$, hence the same argument gives

$$m(\langle D|s\rangle) = m(\langle D|s_k\rangle) \geq m(\langle D|s_{k-1}\rangle) \geq \dots$$
$$\geq \dots \geq m(\langle D|s_1\rangle) > m(\langle D|s_+\rangle)$$

strict inequality

This gives the conclusion, since then $m(\langle D\rangle) = m(\langle D|s_+\rangle)$. (There cannot be cancellations involving the term in $\langle D|s_+\rangle$ with the least possible exponent).

Proposition: Let D be a connected diagram with n crossings. Then

$$|s_+ D| + |s_- D| \leq n+2$$

If D is alternating, then equality holds.

Proof: We prove the first statement by induction on n, the case $n=0$ being obvious. Let D be a connected diagram with $n+1$ crossings. We choose one of them to define the desingularised diagrams D_+, D_- (where only the chosen crossing has been desingularised). At least one of D_+, D_- is connected. Say D_+ is connected. D_+ has n crossings, hence by induction

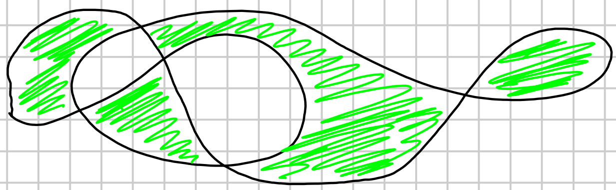
$$|s_+ D_+| + |s_- D_+| \leq n+2$$

Now $s_+ D_+ = s_+ D$, while $|s_- D_+| = |s_- D| \pm 1$

Thus $|s_+ D| + |s_- D| = |s_+ D_+| + |s_- D_+| \pm 1 \leq$

$$\leq n+2 \pm 1 \leq n+3 = (n+1)+2$$

Suppose now D is alternating. Then the components of $s_+ D$, $s_- D$ are the boundary components of the regions defined by $D \subseteq S^2$, considered as a 4-valent graph.



By the Euler-Poincaré formula for S^2,
$V - E + R = 2$. In our case
$V = n = \#$ crossings, $E = 2n$ since the graph is 4-valent, $R = |s_+ D| + |s_- D|$, hence

$$2 = n - 2n + |s_+ D| + |s_- D| \Rightarrow |s_+ D| + |s_- D| = n+2.$$

Recall that the breadth of a polinomial f
$br(f) = M(f) - m(f)$.

Theorem: Let D be a connected diagram of a link L with n crossings.
① $br(V(L)) \leq n$
② If D is reduced and alternating, then $br(V(L)) = n$

As a Corollary, we have the 1st Tait Conjecture

Proof: Of course $br(V(L)) = br(\langle D \rangle)$, since $V(L) = t^{\alpha} \cdot \langle D \rangle$. Thus

$$br(V(L)) = br(\langle D \rangle) = M(\langle D \rangle) - m(\langle D \rangle)$$
$$\leq \frac{n + 2|s_- D| - 2}{4} - \left(-\frac{n + 2|s_+ D| - 2}{4}\right) =$$
$$= \frac{n - 2 + |s_- D| + |s_+ D|}{2} \leq \frac{n - 2 + n + 2}{2} = n$$

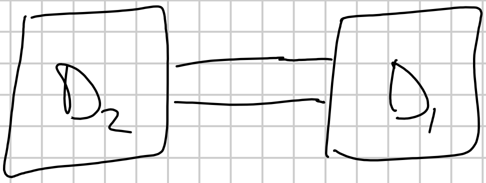
This proves ①. In order to prove ②, we only need to observe that the inequalities are equalities if D is reduced and alternating, since reduced + alternating $\Longrightarrow$ adequate.

Improved statements

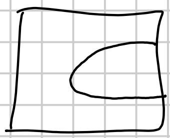
non-alternating

1. If D is a connected, strongly prime diagram of a link L with n crossings, then $br(V(L)) < n$.
In particular, if L admits an alternating diagram, then all minimal diagrams for L are alternating. (At least if L is prime).

Strongly prime



$\Longrightarrow$ D_1 or D_2 is just



2. Let D be a connected + adequate diagram of a link L. Then D minimizes the number of negative crossings in connected diagrams of L. The same for − adequate diagrams (with positive crossings rather than negative ones).

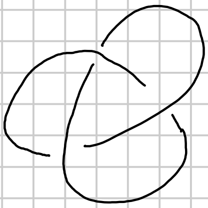
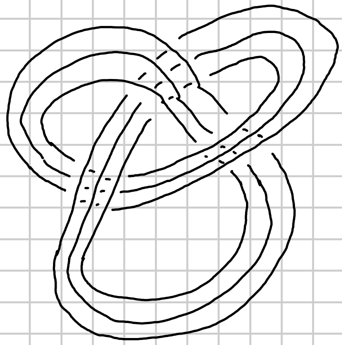
Therefore: • Adequate diagrams are minimal.

• If D, D' are adequate diagrams of the same link, then $wr(D) = wr(D')$

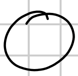
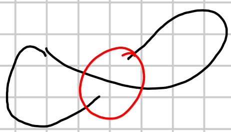
(Second Tait conjecture for alternating knots).

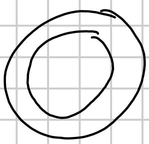
All these statements are proved e.g. in Lickorish's book.

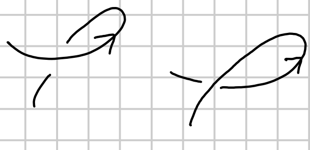
In order to prove (2), one compares diagrams D, where D is + adequate, and E, which is generic.
In fact, one takes the asymptotics of $V(D^2)$ and $V(E^2)$, where D^2, E^2 are obtained by adding parallel copies of the diagrams D, E.

$D =$  $D^3 =$ 

If D and E define the same link, D^2, E^2 may define distinct links!

$D =$  $E =$ 

$D^2 =$  $E^2 =$  Hopf link!

Starting from D, E, one adds 

to define D', E' s.t. corresponding components of D', E' have the same writhe. Then D', E' can be obtained one from the other by type $\underline{II}$, $\underline{III}$ Reidemeister $\Longrightarrow (D')^2 \simeq (E')^2$

Thus $V((D')^2) = V((E')^2)$. This gives an equality $F(t,r) = G(t,r)$, where F, G are classical polynomials of degree 2 w.r.t. r.

By comparing the coefficients of r^2 in F, G, one concludes the proof of 2.