The Jones polynomial

Recall that $P(\ell, m) \in \mathbb{Z}[\ell^{\pm 1}, m^{\pm 1}]$ is the HOMFLY polynomial of links.

We define the Jones polynomial $V(L)$ of a link L by setting

$$V(L) = P(L)(i t^{-1}, i(t^{-\frac{1}{2}} - t^{\frac{1}{2}}))$$

$$\cap$$

$$\mathbb{Z}[t^{\pm \frac{1}{2}}]$$

We have for free the existence of $V(L)$. We now describe a diagrammatic approach to $V(L)$, which could have been defined via a skein relation.

Theorem (definition of $V(L)$ via a skein relation): $V(L)$ is the unique polynomial invariant of links such that:

① $V(O) = V(\text{unknot}) = 1$

② $$t^{-1} V(D_+) - t V(D_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V(D_0) = 0$$

where, as usual, D_+, D_-, D_0 are (diagrams of) links which differ one from the other only in a neighborhood of a crossing.

Proof: We know that skein formulae + normalization on the unknot uniquely define a polynomial function on links (if any). We only need to prove that $V(L)$ indeed satisfies the normalization (which is obvious), and the skein formula. Starting from

$$\ell P(D_+) + \ell^{-1} P(D_-) + m P(D_0) = 0$$

and setting $\ell = i t^{-1}$, $m = i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})$, we get

$$i t^{-1} V(D_+) + (i t^{-1})^{-1} V(D_-) + i(t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V(D_0) = 0$$

$$\cancel{i} t^{-1} V(D_+) - \cancel{i} t V(D_-) + \cancel{i}(t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V(D_0) = 0$$

as desired. $\square$

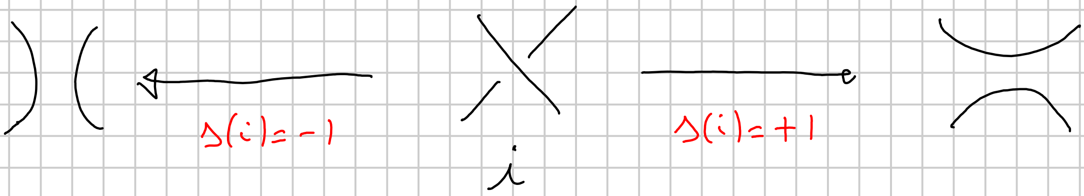
THE KAUFFMAN BRACKET

We describe the Kauffman Bracket (which is a polynomial invariant of diagrams) as a STATE SUM INVARIANT.

For every diagram D, a STATE s

for D is a map $s : \{\text{crossings of } D\} \to \{\pm 1\}$

Then, sD is the diagram of the unlink obtained by desingularizing crossings as follows



This is well-defined regardless of any orientation of D (but it uses the orientation of $\mathbb{R}^2$)

$s(i) = +1 \iff$ the overarc turns left

We denote by $|sD|$ the number of components of the link represented by sD.

For every state s on D we set

$$\langle D \mid s \rangle = \left(t^{-\frac{1}{4}}\right)^{\sum_i s(i)} \cdot \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^{|sD| - 1}$$

Finally, the Kauffman bracket of D is

$$\langle D \rangle = \sum_s \langle D \mid s \rangle .$$

It can be proved (but it is unnecessary for us, since we already know that the Jones polynomial is invariant) that $\langle D \rangle$ is invariant w.r.t. II and III Reidemeister moves (but not w.r.t. I Reidemeister move).

Definition: If D is a diagram, the writhe of D, denoted by $w(D)$, is the sum of the signs of all the crossings of D (here D must be oriented!).

-1 $\qquad\qquad$ $+1$

Example: Adding  to a diagram changes w by subtracting -1, hence $w(D)$ is NOT invariant w.r.t. I Reidemeister move.

Theorem: Let D be a diagram of L.

Then $$V(L) = \left(-t^{-\frac{1}{4}}\right)^{-3w(D)} \langle D \rangle$$

Proof.: In the literature, one usually checks that the right-hand side is invariant w.r.t. I, II, III Reidemeister moves, and then defines the Jones polynomial via this formula. Here we check equality by checking that the right-hand side

$$Z(L) = \left(-t^{-\frac{1}{4}}\right)^{-3w(D)} \langle D \rangle$$

satisfies the skein relation and the normalisation of $V(L)$. Normalisation is obvious. We look at diagrams as follows

D_+ $\qquad$ D_- $\qquad$ D_0 $\qquad$ $\bar{D}$

Let the crossing I am looking at be the 0-th crossing. Any state s for D_+ or D_- defines a state $\bar{s}$ on D_0, $\bar{D}$.

$$\langle D_+\rangle = \sum_{s(0)=+1} \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 0} s(i)} \left(-t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right)^{|sD_+|-1} +$$

$$\sum_{s(0)=-1} \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 0} s(i)} \left(-t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right)^{|sD_+|-1}$$

$$= \sum_{s(0)=+1} \left(t^{-\frac{1}{4}}\right)^{\overset{=s(0)}{1}} \cdot \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 1} \bar{s}(i)} \left(-t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right)^{|\bar{s}D_0|-1} +$$

$$+ \sum_{s(0)=-1} t^{\frac{1}{4}} \cdot \left(t^{-\frac{1}{4}}\right)^{\sum_{i\geq 1} \bar{s}(i)} \left(-t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right)^{|\bar{s}\bar{D}|-1}$$

$$= t^{-\frac{1}{4}} \langle D_0\rangle + t^{\frac{1}{4}} \langle \bar{D}\rangle$$

$$w(D_+) = w(D_0)+1 \qquad w(D_-) = w(D_0)-1$$

$$\left(-t^{-\frac{1}{4}}\right)^{-3w(D_+)} \langle D_+\rangle = \left(-t^{\frac{3}{4}}\right)\left(-t^{-\frac{1}{4}}\right)^{-3w(D_0)} \langle D_+\rangle =$$

$$= \left(-t^{-\frac{1}{4}}\right)^{-3w(D_0)} \cdot \left[t^{-\frac{1}{4}}\langle D_0\rangle + t^{\frac{1}{4}}\langle \bar{D}\rangle\right]\left(-t^{\frac{3}{4}}\right) =$$

$$= \left(-t^{-\frac{1}{4}}\right)^{-3w(D_0)} \cdot \left[-t^{+\frac{1}{2}}\langle D_0\rangle - t\langle \bar{D}\rangle\right]$$

i.e. $V(D_+) = -t^{\frac{1}{2}} V(D_0) - t\left(-t^{-\frac{1}{4}}\right)^{-3w(D_0)} \langle \bar{D} \rangle$

$\Rightarrow t^{-1} V(D_+) = -t^{-\frac{1}{2}} V(D_0) - \left(-t^{-\frac{1}{4}}\right)^{-3w(D_0)} \langle \bar{D} \rangle$

A very similar computation

$$\| $$

$t V(D_-) = -t^{\frac{1}{2}} V(D_0) - \left(-t^{-\frac{1}{4}}\right)^{-3w(D_0)} \langle \bar{D} \rangle$

$$t^{-1} V(D_+) - t V(D_-) = \left(-t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right) V(D_0)$$

Proposition: If L is a link with mirror image $\bar{L}$, then $V(\bar{L})(t) = V(L)(t^{-1})$.

Proof.: It follows from the similar property for P.

Corollary: The Jones polynomial can detect chirality (and, in fact, it can be used to show the trefoil is chiral).

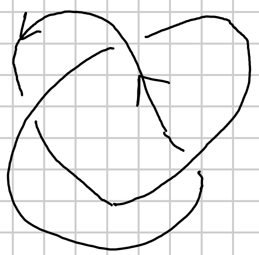
Open question: Let K be a knot with $V(K) = 1$. Is K the unknot?

Remember there are many knots with trivial Alexander-Conway polynomial.

Heuristically, the Jones polynomial is more powerful at distinguishing knots than the Alexander polynomial. (But it is less understood in terms of classical algebraic topology).

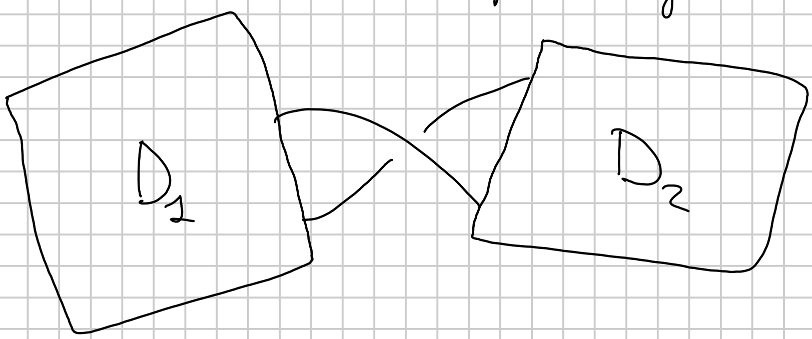
Alternating links

An alternating diagram is a diagram of a link in which every undercrossing is followed by an overcrossing, and viceversa.

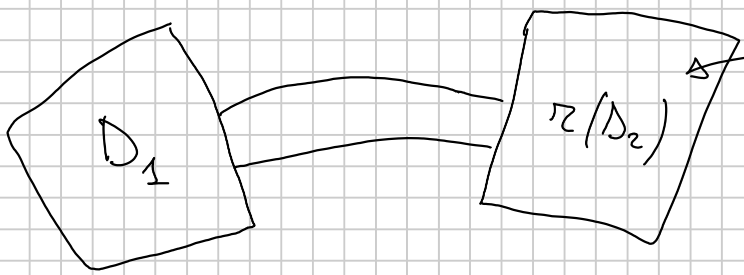


is alternating

A link is alternating if it admits an alternating diagram.
An alternating diagram is reduced if no situation as the following one appears:

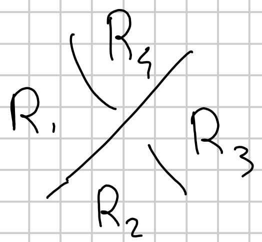


Here, by rotating D_1 or D_2 I can obtain a new diagram of the same link:

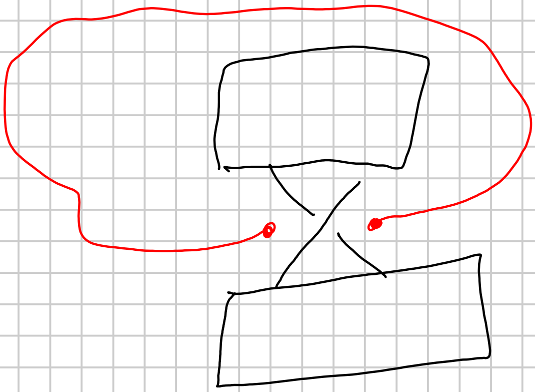


obtained by rotating D_2

A non-reduced diagram is a diagram where at one crossing, the same region of the complement of D appears twice (necessarily in non-adjacent places)



If $R_1 = R_3$ or $R_2 = R_4$ the diagram is not reduced



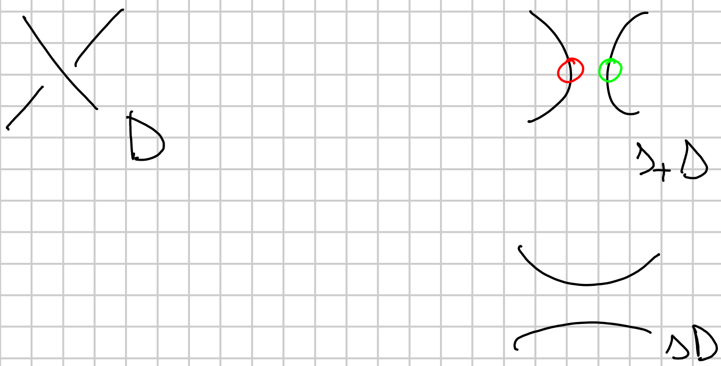
Tait Conjecture (Theorem): Let D be a connected (i.e. non-split) reduced alternating diagram of a link L. Then D realises the crossing number of L, i.e. any other diagram of L has at least the same number of crossings as D. In other words, D minimizes the number of crossings among all the diagrams of L.

Let D be a diagram. Then D is

+ - adequate if $|s_+ D| > |sD|$

for any state s which differs from s_+ at exactly one crossing, where $s_+(i) = +1$ $\forall$ crossing i.

D is $+$-adequate if the same component of s_+D does not abut itself at any former crossing of D, i.e. the components of s_+D appearing at any former crossing of D are distinct.

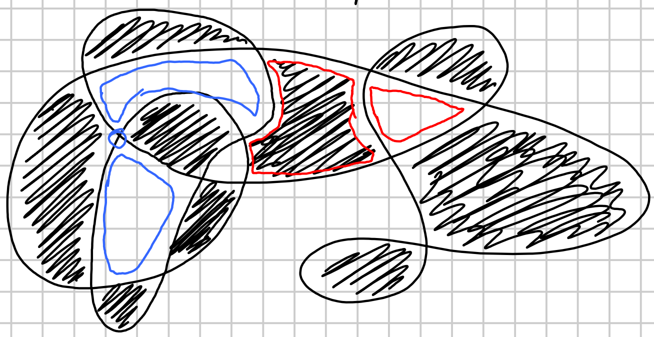


If ○ ○ are distinct, $|sD| = |s_+D| - 1$

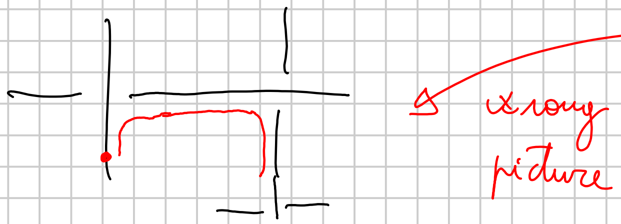
A similar condition defines $-$ adequate diagrams, that are the ones for which $|s_-D| > |sD|$ $\forall s$ obtained by altering s_- exactly at one crossing, $s_-(i) = -1$ $\forall i$.

Lemma: Let D be a non-split reduced alternating diagram. Then D is adequate, i.e. it is both $+$ adequate and $-$ adequate.

Proof: Colour the complementary regions of D in a checkerboard pattern.



If $s(i)=+1$, at the i-th crossing the overarc turns left, while the underarc turns right. The fact D is alternating implies that, in s_+D if you start from a point right before an overpass, you always turn right. left



Thus, the components of s_+D are exactly the circles bounding the black (or white) regions, and the same applies for s_-D.
Since the diagram is reduced, this means that the components of s_+D appearing at any single crossing are distinct, i.e. D is $+$ adequate. The same for s_-D.