Teoria dei Nodi 18/4/19

$\mathcal{S}, \mathcal{S}'$ systems of spheres giving prime decompositions

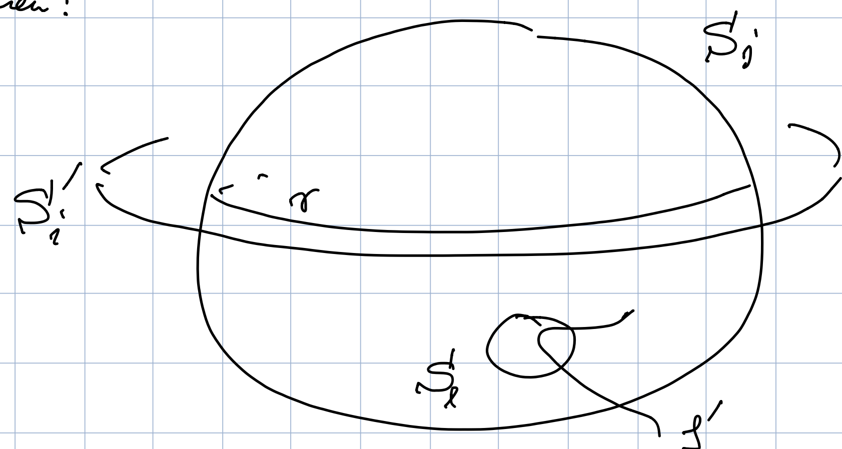
Assume every $S \in \mathcal{S}$ meets $\mathcal{S}'$
every $S' \in \mathcal{S}'$ meets $\mathcal{S}$
+ there is some transverse intersection

Aim: reduce transverse intersection.

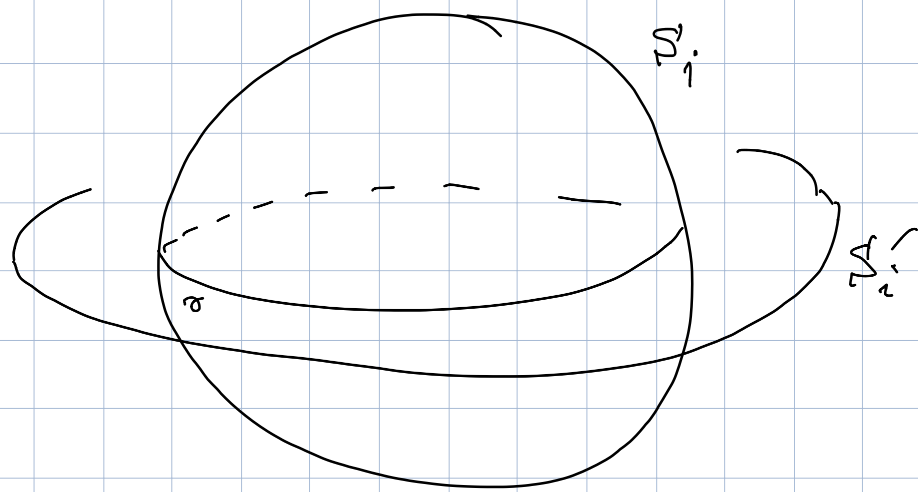
Take $S_i' \in \mathcal{S}'$ with $S_i' \cap \mathcal{S} \neq \emptyset$.
Take $\gamma \in S_i'$ innermost circle in S_i'
$\Rightarrow$ $\gamma = \partial \Delta$, $\Delta \subset S_i'$, $\gamma \subset S_i' \cap S_j$, $\Delta \cap \mathcal{S} = \gamma$.

Let B_j be component of $S^3 \setminus S_j$ with $\Delta \subset B_j$.

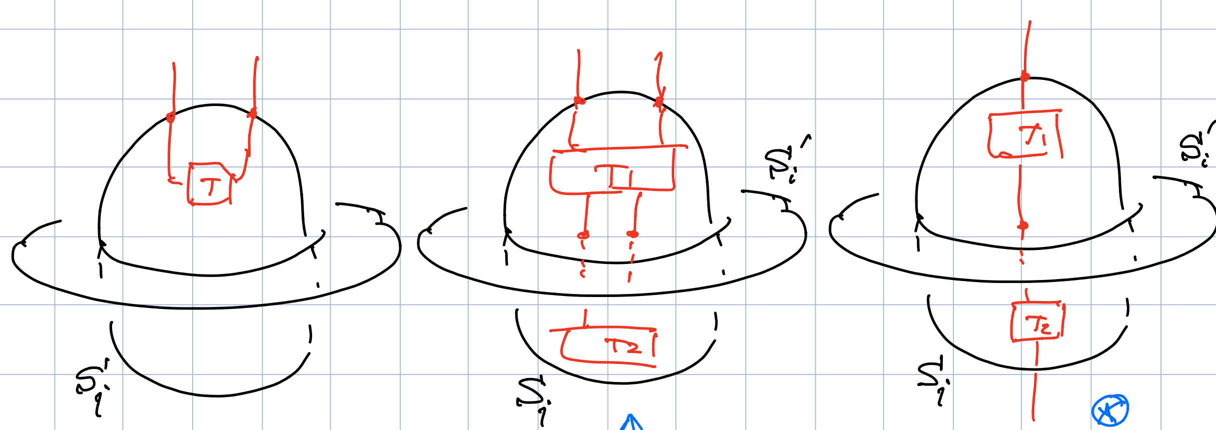
It could happen that B_j contains some other S_l
but then:

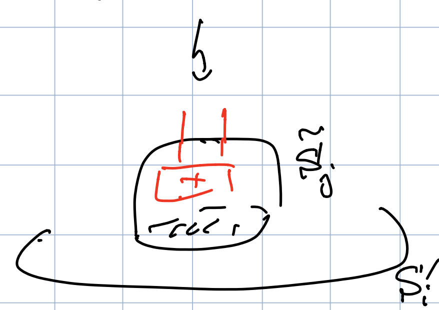


reduce to S_e $\rightsquigarrow$ wlog assume B_j contains no S_l.



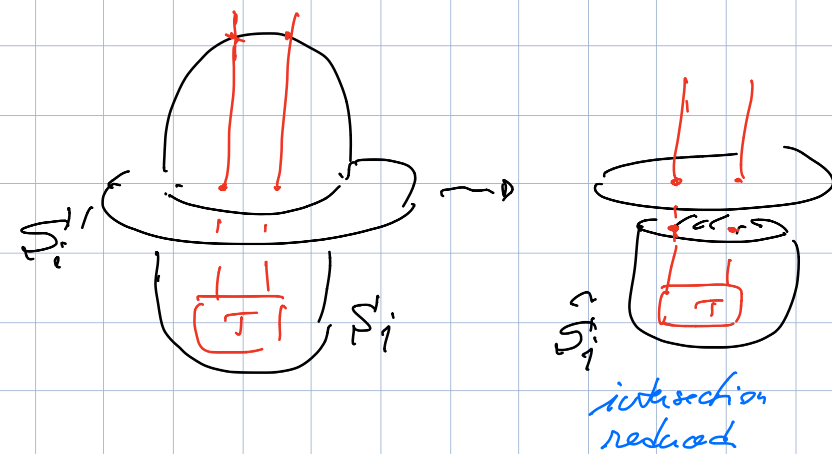
Recall both S_j and S_i' cross K twice: so 3 cases:



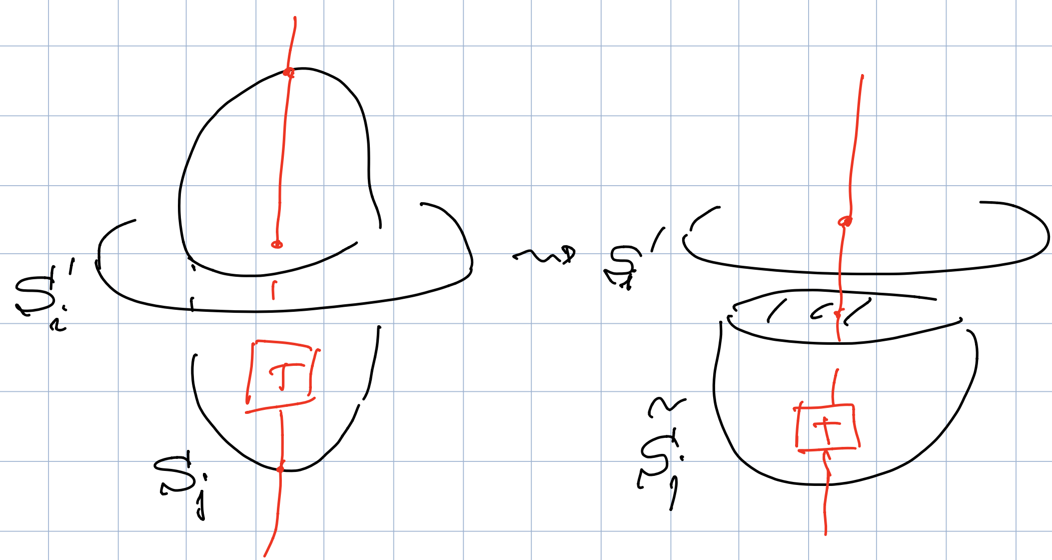


intersection reduced

since S_i' gives splitting into non-trivial links $\Rightarrow T_2$ non-triv.
$\Rightarrow T_1$ must be trivial because S_j splits into primes



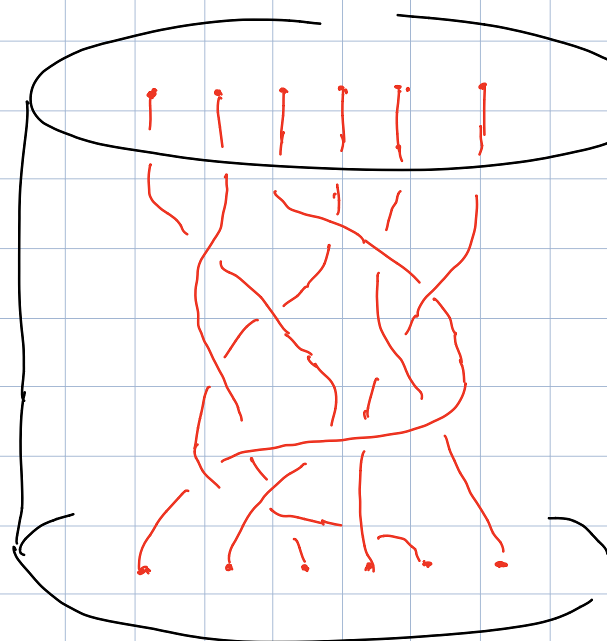
⊗ Since S_j splits into primes must have T_1 or T_2 trivial



Braids and representation of links via braids.

$p_1, \dots, p_n \in \mathrm{int}(D^2)$ fixed pts

Def 1: $B_n = \{ T \subset D^2 \times [0,1] : T \cong \underbrace{[0,1] \sqcup \dots \sqcup [0,1]}_{n}$,
$\partial T = \{p_1, \dots, p_n\} \times \{0,1\}$ s.t.
if $\pi : D^2 \times [0,1] \to [0,1]$ then
$\pi|_\alpha$ monotonic for all $\alpha \subset T \} \big/$ isotopy through such T's



Def 2: $\tilde{B}_n = \{ h : D^2 \to D^2$ homeo : $h|_{\partial D^2} = id_{\partial D^2}$
$h(\{p_1, \dots, p_n\}) = \{p_1, \dots, p_n\} \} \big/$ isotopy through such h's

Actually $B_n = \tilde{B}_n$:

$\tilde{B}_n \longrightarrow B_n$

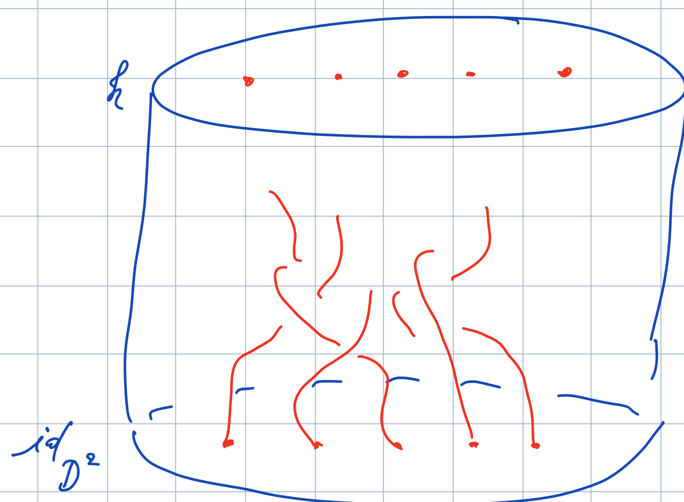
$h: D^2 \to D^2 \quad h|_{\partial D^2} = id \quad h(\{p_1, \dots, p_n\}) = \{p_1, \dots, p_n\}$

$\Rightarrow$ h isotopic to id_{D^2} rel ∂D^2

$\Rightarrow \exists (h_t)_{t \in [0,1]} \quad h_0 = id \quad h_1 = h$

(of course : h_t does not leave $\{p_1, \dots, p_n\}$ invariant)

$T = \{ h_t(\{p_1, \dots, p_n\}) \times \{t\} : t \in [0,1] \}$



$B_n \rightsquigarrow \tilde{B}_n \quad$ take $T \subset D^2 \times [0,1]$

$\Rightarrow T \cap (D^2 \times \{t\}) = \{p_1^{(t)}, \dots, p_n^{(t)}\} \times \{t\}$

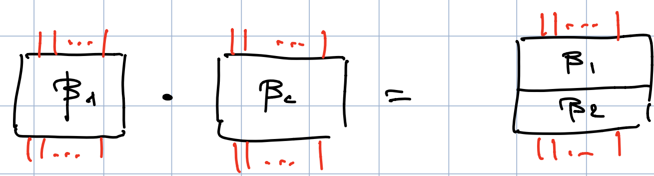
$p_j^{(0)} = p_j^{(1)} = p_j \quad \Rightarrow \exists (h_t)_{t \in [0,1]}$

$h_0 = id \quad h_t|_{\partial D^2} = id \quad h_t(\{p_1, \dots, p_n\}) = \{p_1^{(t)}, \dots, p_n^{(t)}\}$

(promote isotopy to ambient isotopy)

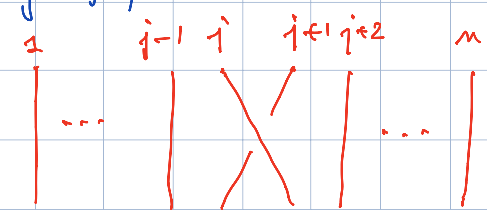
$[T] \rightsquigarrow [h_1] \in \tilde{B}_n$.

Group structure on B_n :



<u>Def</u>: $\sigma_j \in B_n \quad j = 1, \dots, n-1$

$\sigma_j =$



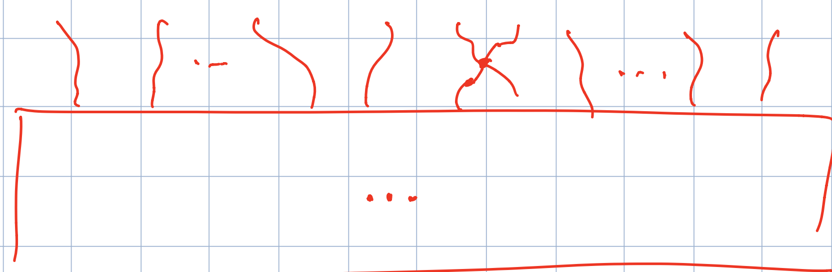
<u>Thm</u>: $B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1, \dots, n-2 \rangle$

<u>Proof</u>: $\sigma_1, \dots, \sigma_{n-1}$ generate (use geometric def. of B_n).

Take β braid; up to isotopy there is a tallest crossing

$\beta =$



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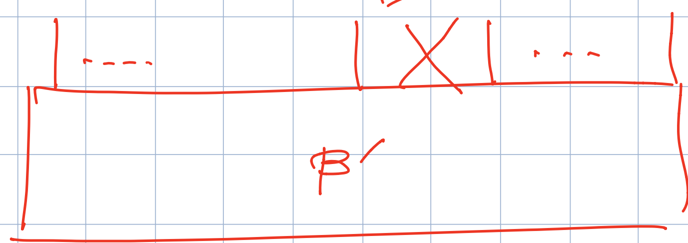
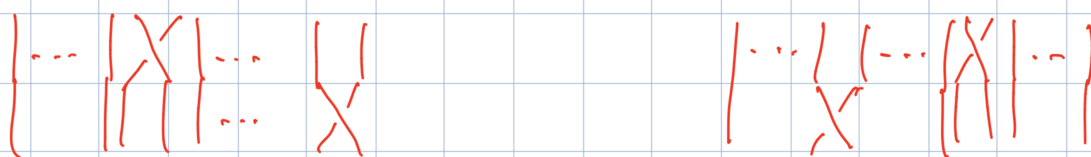
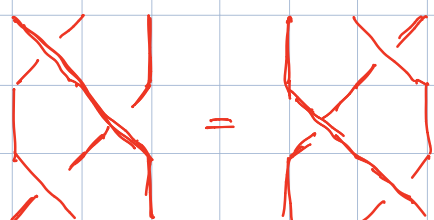


$\Rightarrow \beta = \sigma_i^{\pm 1} \cdot \beta'$.

the relations hold true:

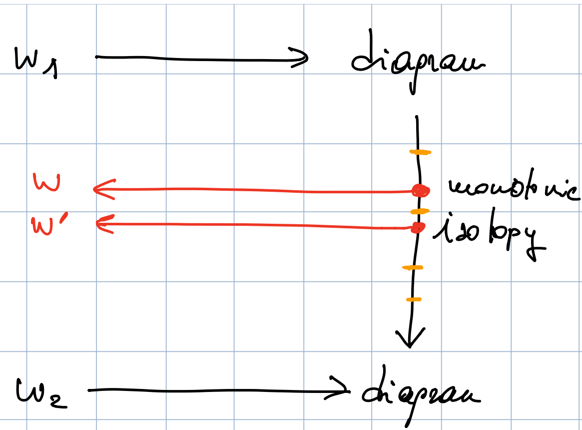




it's R_{III} move

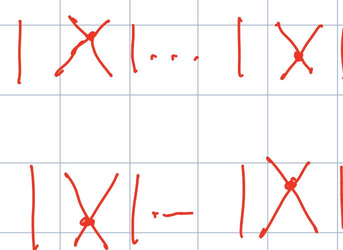
Relations suffice: take words w_1, w_2 in $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$
suppose they give the same geometric braid
i.e. their diagrams are related by planar isotopy
through monotonic diagrams + Reidemeister moves
through monotonic diagrams.

Notice: if a monotonic diagram has all crossings
at different heights then it determines a word in $\sigma_1^{\pm 1} \dots \sigma_n^{\pm 1}$



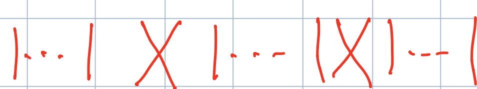
• elementary catastrophes
||
times at which two crossings are at same height (only two)

+ nonmonotonic R moves



effect: some $\sigma_i \sigma_j$ replaced by $\sigma_j \sigma_i$ implied by relations.

type I
height change



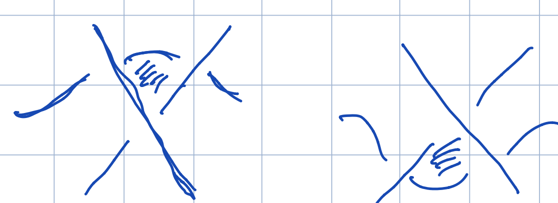
R_I: never happen

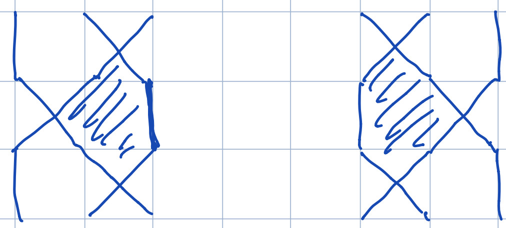
R_{II}:

some $\sigma_i^{\pm 1} \sigma_i^{\mp 1}$ gets cancelled

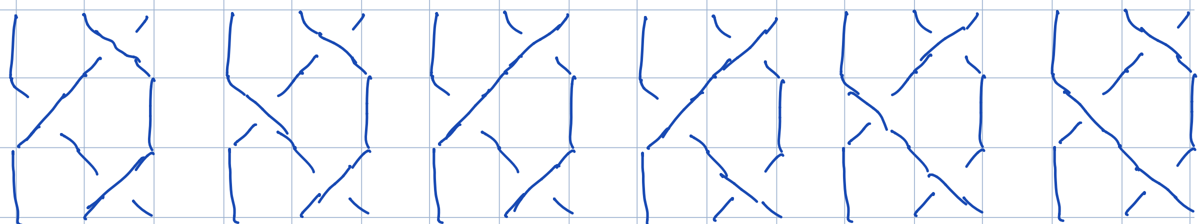
R_{III}

two possible examples:

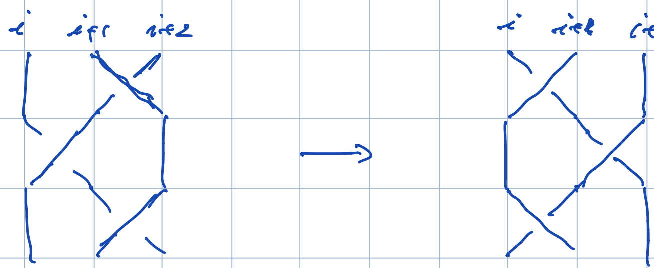


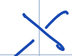


for each of them: 6 config's of crossings:



Fact all 12 cases translate into a replacement of a word of length 3 by another one implied by $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. E.g.



$\sigma_i :=$ 

$$\sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \qquad \qquad \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i$$

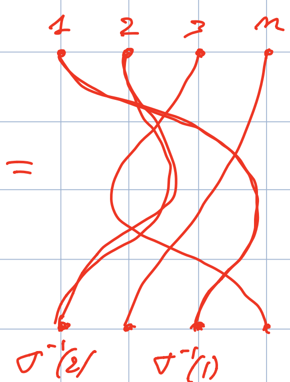
$$\sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i \cancel{\sigma_{i+1}} \cancel{\sigma_i} = \cancel{\sigma_{i+1}} \cancel{\sigma_i} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}$$

Remark: there is a natural $B_n \twoheadrightarrow \mathfrak{S}_n$

$$\mathfrak{S}_n = \langle \tau_1, \dots, \tau_{n-1} \mid \tau_i \tau_j = \tau_j \tau_i \quad |i-j| \geq 2, \quad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \tau_i^2 = 1 \rangle$$

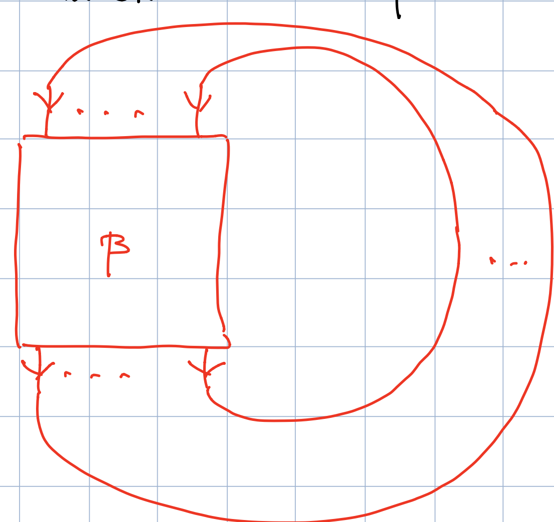
$\tau_j = (j, j+1)$

given by $\sigma_i \longmapsto \tau_i$

$\beta =$ 

$\beta \longmapsto \sigma$

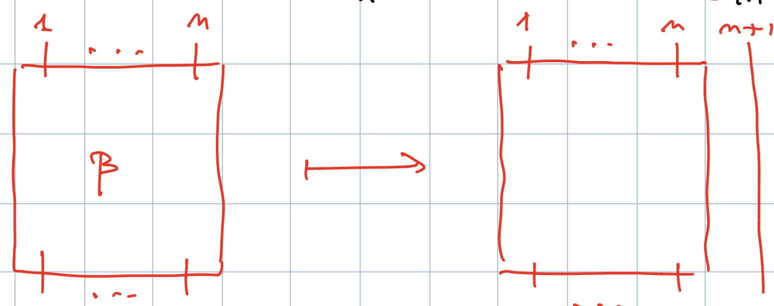
Def: if $\beta \in B_n$ is a braid I call closure of β the oriented link $\hat{\beta}$



Thm (Alexander): every oriented link is $\hat{\beta}$ for some n and some $\beta \in B_n$.

(several proofs via bridge presentation + Seifert surface...)

Rem: there is a natural inclusion $B_n \longrightarrow B_{n+1}$



Def: Markov moves on braids:

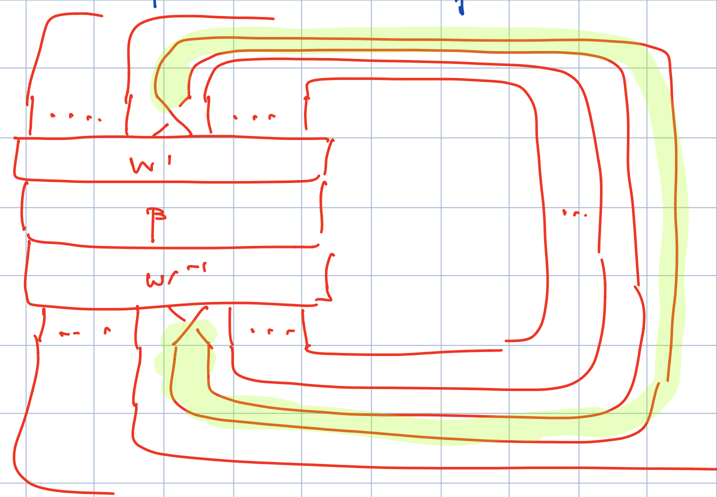
- conjugation : $\beta \in B_n \longmapsto w\beta w^{-1} \in B_n$
- stabilization $\beta \in B_n \subset B_{n+1} \longmapsto \beta\sigma_n^{\pm 1} \in B_{n+1}$

We say $\beta \in B_n, \beta' \in B_{n'}$ are Markov equivalent if related by combination of aboves and inverses.

Remark: Markov equivalent braids have isotopic closures:

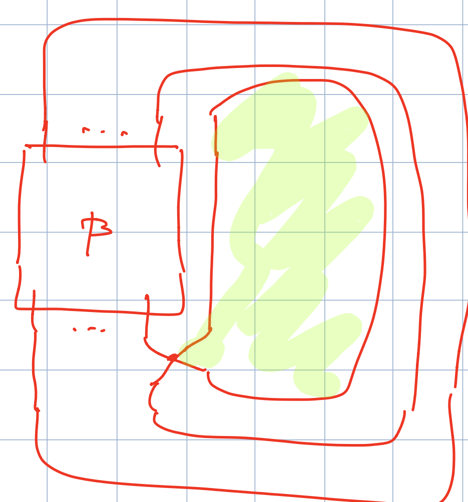
conjugation: $\beta \longrightarrow w\beta w^{-1}$ $\qquad w = \sigma_j w'$

$\widehat{w\beta w^{-1}} =$



$\cong \hat{\beta}$

stabilization: $\widehat{\beta\sigma_n^{\pm 1}} =$



R_I applies

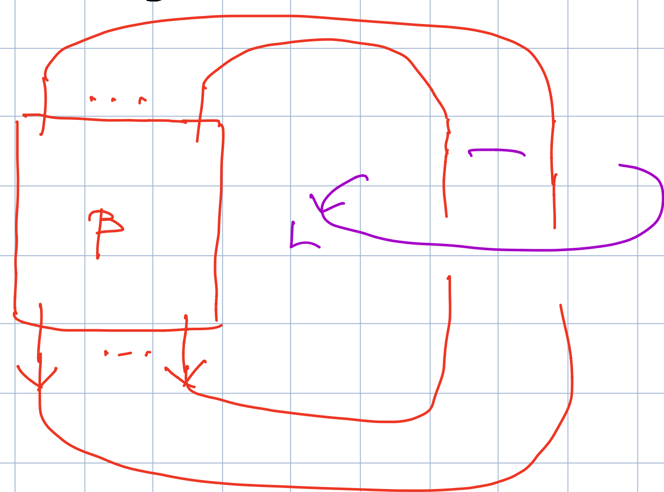
$\cong \hat{\beta}$

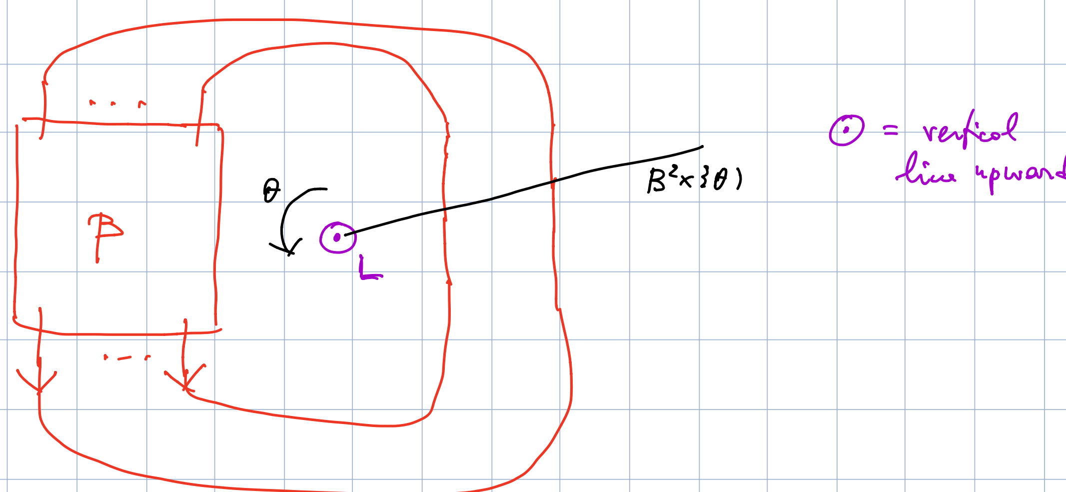
Thm (Markov): $\beta \in B_n$, $\beta' \in B_{n'}$

$\hat{\beta} \cong \hat{\beta'} \iff \beta, \beta'$ are Markov equivalent.

Proofs after H. Morton

Def: if $\beta \in B_n$ I call complete closure of β the oriented link $\hat{\beta} \cup L$ where L is the axis of $\hat{\beta}$ oriented s.t. $\hat{\beta}$ wraps positively around it:



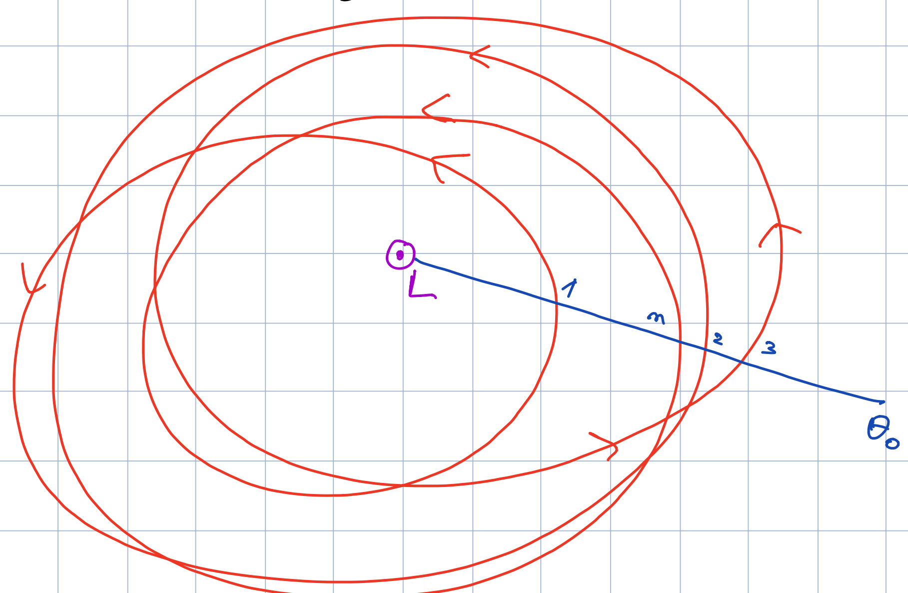


Note L = trivial knot $\Rightarrow S^3 \setminus L = B^2 \times S^1_\theta$ and $\hat{\beta} \subset B^2 \times S^1_\theta$ in such a way that the associated projection $p_L : S^3 \setminus L \to S^1_\theta$ restricts to be strictly increasing to each component of $\hat{\beta}$. Moreover $B^2 \times \{\theta\}$ = closed disc bounded by L.

Def: an oriented link $K \cup L$ is braided if L is trivial and $S^3 \setminus L \cong B^2 \times S^1$ so that the associated projection p_L restricts to be strictly increasing on each component of K.

+ compatible orientation....

Remark: every braided link determines a braid up to conjugation:



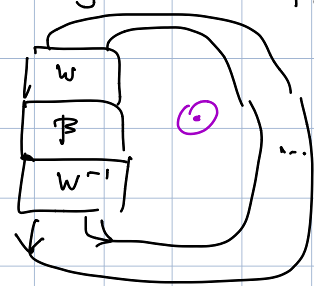
- choose level disc $B^2 \times \{\theta_0\}$ where to cut
- label points $K \cap (B^2 \times \{\theta_0\})$ as $1, \dots, n$

$\rightsquigarrow$ braid

both choices made only give conjugation.

Prop: $\beta \in B_n$, $\beta' \in B_{n'}$ have isotopic complete closures iff they are conjugate in B_n.

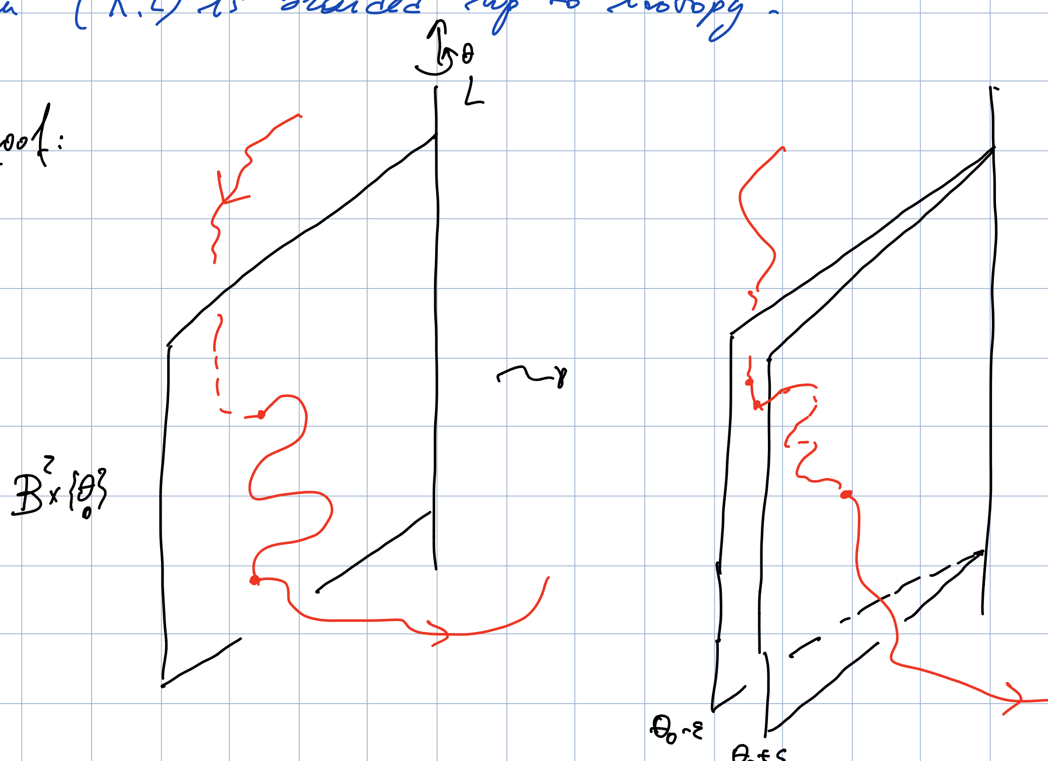
"if" easy



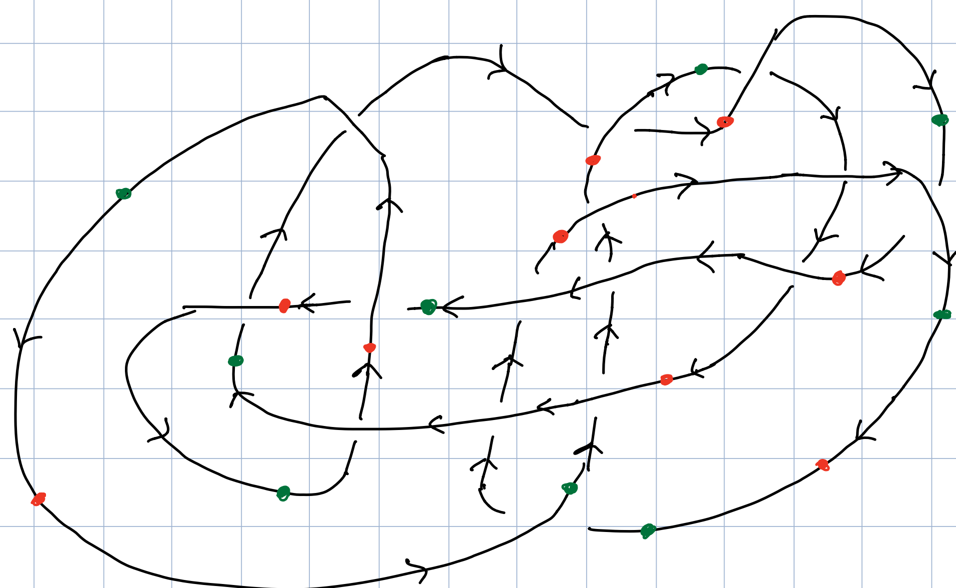
"only if": Remark just made.

Prop: if (K, L) is oriented link with L trivial and $S^3 \setminus L \cong B^2 \times S^1$ s.t. $p_L|_J$ non-decreasing and non-constant for all components J of K then (K, L) is braided up to isotopy.

Proof:



Def: if K is an oriented link diagram I call choice of overarcs the choice of point on K S ("start") E ("end") s.t. on K each arc from S to E contains overcrossings only and each arc from E to S contains undercrossings only (possibly none):



S = "start"

E = "end"