Knot theory 2/4/2019

Let K be a knot, $E(K) = S^3 \setminus K$.

$\pi_1(E(K)) = G = \langle s_1, \dots s_n \mid r_1, \dots, r_k \rangle$.

$\varphi : F_n \to G \to G/G' = H_1(E(K))$, which extends linearly to

$\varphi : \mathbb{Z}[F_n] \longrightarrow \mathbb{Z}[t, t^{-1}]$.

Theorem: The matrix with n rows and k columns given by $A_{ij} = \varphi\left(\dfrac{\partial r_j}{\partial s_i}\right)$ is a presentation matrix for the UNREDUCED Alexander module of K, i.e. $H_1(\widetilde{E(K)}_\infty, p^{-1}(x_0))$, $p : \widetilde{E(K)}_\infty \to E(K)$.

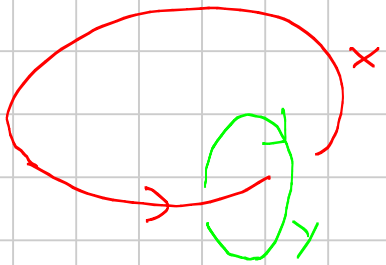
TORUS KNOTS

Let $p, q \in \mathbb{Z} \setminus \{-1, 0, 1\}$ with $(p, q) = 1$, and let $K_{p,q}$ be the (p, q)-torus knot.

Proposition: $\pi_1(E(K_{p,q})) = \langle x, y \mid x^p y^{-q} \rangle$.

Moreover, $\varphi(x) = t^q$, $\varphi(y) = t^p$.

Proof.:



Let the red loop be the core of the solid torus T such that $K_{p,q} \subseteq \partial T$.

Then $T = \overline{N(\text{red arc})}$, and if

$T' = \overline{N(\text{green arc})}$, then

$S^3 = T \cup T'$, $T \cap T' = \partial T = \partial T'$.

We now use Van Kampen on the decomposition

$$E(K_{p,q}) = S^3 \setminus K_{p,q} = (T \setminus K_{p,q}) \cup (T' \setminus K_{p,q}).$$

$$\pi_1(T \setminus K_{p,q}) = \pi_1(T) = \langle x \rangle \cong \mathbb{Z}$$

$$\pi_1(T' \setminus K_{p,q}) = \pi_1(T') = \langle y \rangle \cong \mathbb{Z}$$

$$(T' \setminus K_{p,q}) \cap (T \setminus K_{p,q}) = \partial T \setminus K_{p,q} = \partial T' \setminus K_{p,q}$$

which is an (open) annulus, hence

$$\pi_1((T' \setminus K_{p,q}) \cap (T \setminus K_{p,q})) \cong \mathbb{Z} = \langle [\gamma] \rangle$$

where γ is a loop on ∂T parallel to $K_{p,q}$. Therefore, via the inclusions

$$(\partial T \setminus K_{p,q}) \xrightarrow{i} T \setminus K_{p,q}$$

$$(\partial T' \setminus K_{p,q}) \xrightarrow{i'} T' \setminus K_{p,q}$$

we have $i([\gamma]) = x^p$

$$i'([\gamma]) = y^q.$$

By Van Kampen, we have

$$\pi_1(E(K)) = \langle x, y \mid x^p = y^q \rangle$$

By definition $\varphi(x) = lk(x, K_{p,q}) = q = t^q$

$$\varphi(y) = lk(y, K_{p,q}) = t^p$$

Alexander polynomial of torus knots

Let $r = x^p y^{-q} \in F_2 = F\langle x, y \rangle$ free group over x, y.

$$\frac{\partial r}{\partial x} = \frac{\partial x^p}{\partial x} \cdot \varepsilon(y^{-q}) + x^p \cdot \frac{\partial y^{-q}}{\partial x} =$$

$$= (1 + x + \dots + x^{p-1}) \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} \cdot 1 + x^p(-y^{-1} \dots - y^{-q}) \underbrace{\frac{\partial y}{\partial x}}_{=0} =$$

$$= 1 + x + \dots + x^{p-1}$$

We used $\quad D(g^n) = (1 + g + \dots + g^{n-1}) D(g)$

$$D(g^{-n}) = (-g^{-1} - \dots - g^{-n}) D(g).$$

$$\frac{\partial r}{\partial y} = \frac{\partial (x^p y^{-q})}{\partial y} = \underbrace{\frac{\partial x^p}{\partial y}}_{=0} \varepsilon (y^{-q}) + x^p \frac{\partial y^{-q}}{\partial y} =$$

$$= x^p (-y^{-1} - \dots - y^{-q})$$

Hence, $\varphi\left(\frac{\partial r}{\partial x}\right) = \varphi(1 + x + \dots + x^{p-1}) =$

$$= 1 + t^q + t^{2q} + \dots + t^{(p-1)q} =$$

$$= \frac{t^{pq} - 1}{t^q - 1}$$

$$\varphi\left(\frac{\partial r}{\partial y}\right) = \varphi(x^p(-y^{-1} - \dots - y^{-q})) =$$

$$= t^{pq}(-t^{-p} - \dots - t^{-pq}) =$$

$$= -1 - t^p - t^{2p} \dots - t^{(q-1)p} =$$

$$= -\frac{t^{pq} - 1}{t^p - 1}.$$

Hence, a presentation matrix for the UNREDUCED Alexander module of $K_{p,q}$

is
$$\begin{pmatrix} \dfrac{t^{pq}-1}{t^q-1} \\ \\ \dfrac{t^{pq}-1}{t^p-1} \end{pmatrix}$$

We'd like to compute $\Delta_{K_{p,q}}$, that is the generator of the first elementary ideal of the REDUCED Alexander module, which coincides with the SECOND elementary ideal of the unreduced module.

Hence, we look at the ideal generated by $(m-n+1)$-minors, i.e. $(2-2+1)$-minors.
We know (from other results) that this ideal is principal and generated by

$$\Delta_{K_{p,q}} = G.C.D.\left(\frac{t^{pq}-1}{t^p-1}, \frac{t^{pq}-1}{t^q-1}\right)$$

Since p,q are coprime, hence
$G.C.D.(t^p-1, t^q-1) = t-1$ (these polynomials

have 1 as the unique common root in $\mathbb{C}$).

Moreover, t^p-1 and t^q-1 divide $t^{pq}-1$

Thus

$$t^{pq}-1 = (t-1)(1+t+\dots+t^{p-1})(1+t+\dots+t^{q-1})\cdot\Delta,$$

and $$\frac{t^{pq}-1}{t^p-1} = (1+t+\dots+t^{q-1})\cdot\Delta$$

$$\frac{t^{pq}-1}{t^q-1} = (1+t+\dots+t^{p-1})\cdot\Delta$$

Hence $\Delta = \Delta_{K_{p,q}}(t)$. Thus

$$\Delta_{K_{p,q}}(t) = \frac{t^{pq}-1}{(t^q-1)(1+t+\dots+t^{p-1})} = \frac{(t^{pq}-1)(t-1)}{(t^q-1)(t^p-1)}$$

Corollary: $\text{genus}(K_{p,q}) = \frac{(p-1)(q-1)}{2}$.

Proof: We already know $\leq$.

Now we have $\text{genus}(K_{p,q}) \geq \frac{br(\Delta_{K_{p,q}})}{2} =$

$$= \frac{pq+1-p-q}{2} = \frac{(p-1)(q-1)}{2}.$$

Corollary (Classification of torus knots):

$T_{p,q}$ is isotopic to $T_{p',q'}$ (or to its mirror image) $\Longleftrightarrow \{|p|,|q|\} = \{|p'|,|q'|\}$

Proof.: By looking at the roots of $\Delta_{K_{p,q}}$, we deduce that $\Delta_{K_{p,q}} \doteq \Delta_{K_{p',q'}}$

$\Longleftrightarrow \{|p|,|q|\} = \{|p'|,|q'|\}$

The conclusion follows from the fact that $K_{p,q}$ is INVERTIBLE (same proof as for the trefoil).

Remark: $T_{p,q} \not\simeq T_{p,-q}$ (= mirror of $T_{p,q}$), but the Alexander polynomial cannot detect this fact.

FOX DERIVATIVES AND THE WIRTINGER PRESENTATION

Let K be a knot, and let

$\pi_1(E(K)) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$

be a Wirtinger presentation for $\pi_1(E(K))$ coming from a diagram D for K.

1. For every $i = 1, \dots, n$, also
$$\pi_1(E(K)) = \langle x_1, \dots, x_n \mid r_1, \dots, \widehat{r_i}, \dots, r_n \rangle.$$

2. Each x_i is a meridian, hence $\varphi(x_i) = t$ $\forall i = 1, \dots, n$.

We can now apply the strategy described in the last lessons to get a presentation matrix for the unreduced Alexander module.

(Then, we want to compute the 2nd elementary ideal).

This matrix is $n \times n$. From 1, we can delete one column thus getting an $n \times (n-1)$ matrix. Now I should compute the G.C.D. of all the n $(n-1)\times(n-1)$ minors.

Claim: Instead, I can take just one of these minors, i.e. I can delete also a row, and take the determinant.

Proof of the claim: We prove we can delete the i-th row. $M = E(K)$

$$0 \to H_1(\widetilde{M}_\infty) \to H_1(\widetilde{M}_\infty, \widetilde{M}_0) \to \operatorname{Ker} \varepsilon \to 0$$

$\operatorname{Ker} \varepsilon = (t-1) \subseteq \mathbb{Z}[t,t^{-1}] \cong H_0(\widetilde{M}_0)$.

$\operatorname{Ker} \varepsilon$ is a free $\mathbb{Z}[t,t^{-1}]$-module generated by $t-1$. As a section

$s: \operatorname{Ker} \varepsilon \to H_1(\widetilde{M}_\infty, \widetilde{M}_0)$ I can choose

$(t-1) \to \widetilde{x}_i$ (the lift of the generator x_i starting at the basepoint of $\widetilde{M}_\infty$)

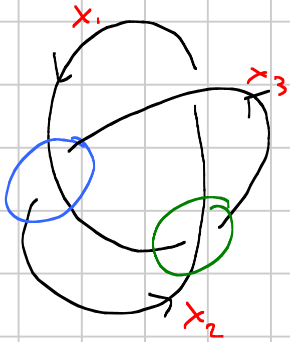
In fact $\partial \widetilde{x}_i$ = endpoint of $\widetilde{x}_i$ − starting point of $\widetilde{x}_i$

$= t-1$

Therefore $H_1(\widetilde{M}_\infty, \widetilde{M}_0) \cong H_1(\widetilde{M}_\infty) \oplus \mathbb{Z}[t,t^{-1}] \cdot \widetilde{x}_i$

Therefore, putting $\tilde{x}_i = 0$ in $H_1(\tilde{M}_\infty, \tilde{M}_0)$ gives $H_1(\tilde{M}_\infty)$, which is equivalent to the Claim.

Example:



$r_1 = x_1 x_3 x_1^{-1} x_2^{-1}$

$r_2 = x_2 x_1 x_2^{-1} x_3^{-1}$

$$\frac{\partial r_1}{\partial x_1} = 1 - x_1 x_3 x_1^{-1} \overset{\varphi}{\longmapsto} 1 - t$$

$$\frac{\partial r_1}{\partial x_2} = - x_1 x_3 x_1^{-1} x_2^{-1} \overset{\varphi}{\longmapsto} -1$$

$$\frac{\partial r_2}{\partial x_1} = x_2 \longmapsto t$$

$$\frac{\partial r_2}{\partial x_2} = 1 - x_2 x_1 x_2^{-1} \overset{\varphi}{\longmapsto} 1 - t$$

$$\Delta = \det \begin{pmatrix} 1-t & t \\ -1 & 1-t \end{pmatrix} = (1-t)^2 + t = t^2 - t + 1$$

In general, every relation in a Wirtinger presentation is of the form $x_i x_j x_i^{-1} x_k^{-1} = r_{ijk}$ and always

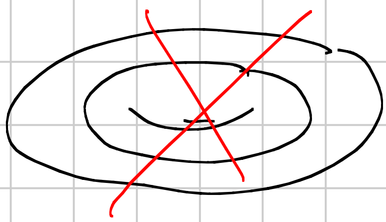
$$\varphi\left(\frac{\partial r_{ijk}}{\partial x_i}\right) = 1 - t \qquad \varphi\left(\frac{\partial r_{ijk}}{\partial x_j}\right) = t, \quad \varphi\left(\frac{\partial r_{ijk}}{\partial x_k}\right) = -1$$

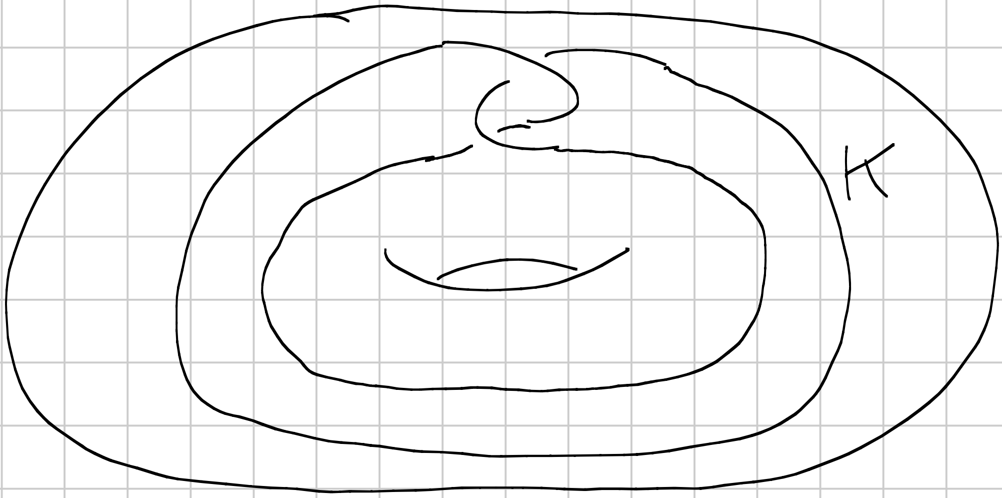
Therefore, just by looking at the diagram I can combinatorially construct an $(n-1)\times(n-1)$ matrix ($n = \#$ crossings) with entries $1-t$, t, -1 to compute the Alexander polynomial.

Satellite knots

Let T be the standard solid torus, i.e. a regular neighborhood of the unknot, and let $K \subseteq \mathring{T}$ be a knot such that

1. K is not isotopic to the core of T.
2. K is not contained in a ball $B \subseteq T$ (equivalently, K meets every meridian disk of T)





Let C be a knot in S^3 with regular neighbourhood T', and let $e : T \longrightarrow T'$ be a diffeomorphism taking the longitude on ∂T to the longitude of $\partial T'$.

Set $K' = e(K)$.

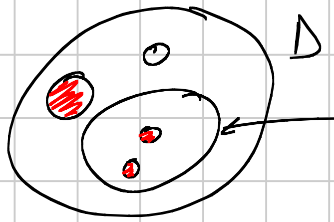
Then K' is a SATELLITE KNOT with COMPANION C, PATTERN K.

Theorem: If C is not trivial, any satellite knot of C is not trivial.

Proof: By contradiction, suppose $K' = e(K)$

is trivial, i.e. $K' = \partial D$, D embedded disk. I can move D transverse to $\partial T'$.

Thus $D \pitchfork \partial T'$ is a collection of circles in $\mathring{D}$. It cannot be empty, otherwise $D \subseteq T' \Rightarrow$ the pattern is trivial in T, a contradiction. I will show we can decrease the number of components of $D \pitchfork \partial T'$, and this gives a contradiction (since I can get $D \pitchfork \partial T' = \phi$ iterating the procedure).



Take an innermost circle in $D \cap \partial T'$, which bounds a disk $\overline{D} \subseteq D$. If $\partial \overline{D} \subseteq \partial T'$ is trivial in $\partial T'$, then it bounds a disk $\overline{\overline{D}} \subseteq \partial T'$. We can lower the number of components of $D \cap \partial T'$ by replacing $\overline{D}$ with $\overline{\overline{D}}$ and pushing.

Hence $\partial \overline{D}$ is not trivial in $\partial T'$.

If $\overline{D} \subseteq T'$ then $\overline{D}$ is a meridian disk disjoint from K', which is not allowed.

If $\overline{D} \subseteq \overline{S^3 \setminus T'}$, then $\partial \overline{D}$ is a longitude of $\partial T'$, and it can be completed by adding an annulus to a disk $\overline{\overline{D}}$ with $\partial \overline{\overline{D}} = C$, hence C is trivial, a contradiction.