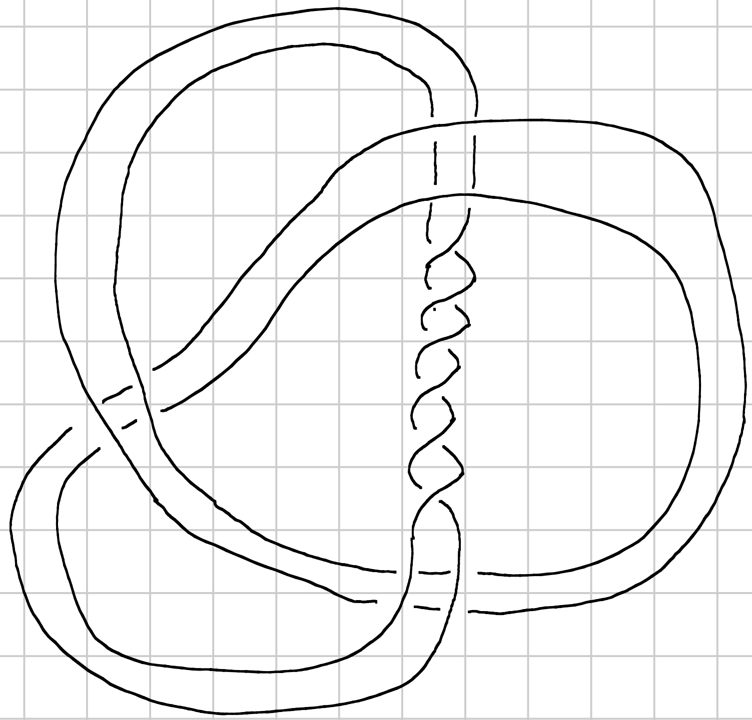
Knot theory 28/3/2019



Proof that this link is not split: see the webpage of the course

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Fix a path-connected space (M, x_0)

$\pi_1(M, x_0) = G$, $H_1(M) = G/G' \cong \mathbb{Z}$

with a preferred generator.

$\tilde{M}_\infty$ = infinite cyclic covering associated to G'

$\Rightarrow \mathrm{Aut}(\tilde{M}_\infty) \cong G/G' \cong \mathbb{Z}$

If $\overline{t} \in \pi_1(M)$ projects onto the generator of $H_1(M)$, we defined an action $\overline{t} \circlearrowright G'/G''$, $G'/G'' \cong H_1(\tilde{M}_\infty)$. The action is by conjugation, and it depends only on the class of $\overline{t}$ in G/G' (exercise). This defines an action

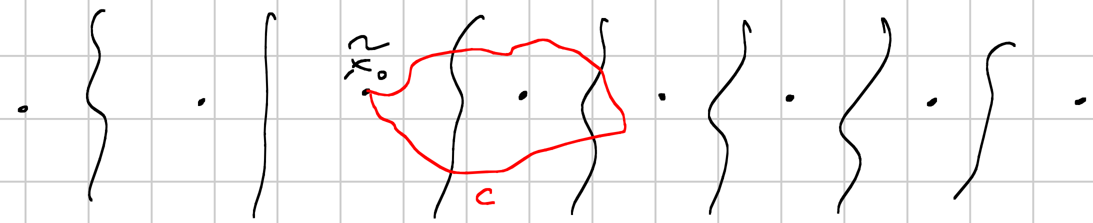
$$\begin{array}{c} G/G' \circlearrowright G'/G'' \\ \cong \\ \mathbb{Z} \end{array}$$

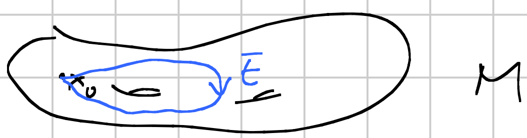
Let $[c] \in H_1(\tilde{M}_\infty) = G'/G''$. What is $\overline{t} \cdot [c]$?

By Hurewicz, after fixing $\tilde{x}_0 \in p^{-1}(x_0)$,

$$p : \tilde{M}_\infty \longrightarrow M,$$

we may suppose c is a loop based at $\tilde{x}_0$.





By definition, if I denote by $\tilde{\gamma}$ the lift of a path γ starting at x_0 with initial point $\tilde{x}_0$, then

$$t\cdot[c] = \left[\widetilde{t * p(c) * t^{-1}}\right] =$$

$$= \left[\tilde{t} * (\tau\circ c) * \tau(\widetilde{t^{-1}})\right] =$$

this ends in $\tau(\tilde{x}_0)$

where τ generates $Aut(\tilde{M}_\infty)$

$$= [\tilde{t}] + \tau_*([c]) + [\tau(\widetilde{t^{-1}})] = \tau_*([c])$$

are one the inverse path of the other

This proves that the action $G/G' \curvearrowright G'/G''$

coincides with the action $\mathbb{Z}[t,t^{-1}] \curvearrowright H_1(\tilde{M}_\infty)$

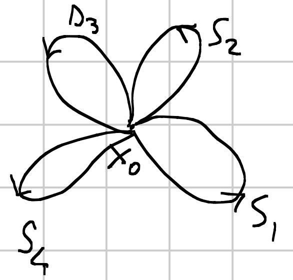
Corollary: The structure of $H_1(\tilde{M}_\infty)$ as a $\mathbb{Z}[t,t^{-1}]$-module only depends on $G=\pi_1(M)$.

We apply this fact to $E(K)$, where K is a knot (so that $H_1(E(K)) \cong \mathbb{Z}$).

Suppose we have a presentation of $G = \pi_1(E(K))$

$$G = \langle s_1, \dots s_n \mid r_1, \dots r_k \rangle.$$

Then I construct a space M with $\pi_1(M) = G$ by taking a 2-dimensional CW-complex with one vertex x_0, n 1-cells (which are necessarily loops), and k 2-cells, which "read" the relations $r_1, \dots, r_k$.



If $r_1 = s_1 s_2 s_1^{-1} s_3$ I attach a 2-cell with boundary given by $s_1 * s_2 * s_1^{-1} * s_3$. By Van Kampen, $\pi_1(M, x_0) = G$.

By Hurewicz, $H_1(M) = \frac{G}{G'} \cong \mathbb{Z}$, hence we can construct $\widehat{M}_\infty$ as usual.

Let $p : \widehat{M}_\infty \to M$ the cyclic covering,

and $\widetilde{M}_0 = p^{-1}(x_0)$ (which is a countable set),

and also fix $\tilde{x}_0 \in p^{-1}(x_0) = \widehat{M}_0$.

We are going to compute the UNREDUCED

Alexander module of M (or of G), that is

$$H_1(\widehat{M}_\infty, \widetilde{M}_0),$$

which again is obviously a $\mathbb{Z}[t, t^{-1}]$-module,

where $t = \tau_*$, $\tau \in Aut(\widetilde{M}_\infty)$ is the positive generator.

(In the focused literature, $H_1(\widehat{M}_\infty)$ is the

REDUCED Alexander module).

Proposition: $H_1(\widetilde{M}_\infty, \widetilde{M}_0) \cong H_1(\widetilde{M}_\infty) \oplus \mathbb{Z}[t, t^{-1}]$.

Hence, $\forall r \in \mathbb{N}$ $E_r(H_1(\widetilde{M}_\infty)) = E_{r+1}(H_1(\widehat{M}_\infty, \widetilde{M}_0))$.

(Warning: things are more complicated

for links).

Proof: We have the exact sequence

$$\begin{array}{ccccccccc} H_1(\tilde{M}_0) & \to & H_1(\hat{M}_\infty) & \to & H_1(\hat{M}_\infty, \hat{M}_0) & \to & H_0(\hat{M}_0) & \xrightarrow{\varepsilon} & H_0(\hat{M}_\infty) \\ \| & & & & & & \| & & \| \\ 0 & & & & & & \mathbb{Z}[t,t^{-1}]\cdot\tilde{x}_0 & & \mathbb{Z} \end{array}$$

ε = augmentation map $\quad \varepsilon(p(t)) = p(1)$

$$\left(\sum a_i (t^i \tilde{x}_0) \to \sum a_i\right)$$

Hence $\operatorname{Ker} \varepsilon = (t-1)$ (augmentation ideal)

Therefore $\operatorname{Ker} \varepsilon$ is a free $\mathbb{Z}[t,t^{-1}]$-module on one generator $\Rightarrow \operatorname{Ker} \varepsilon \cong \mathbb{Z}[t,t^{-1}]$ (as a $\mathbb{Z}[t,t^{-1}]$-module, NOT as a ring).

Thus I have the exact sequence

$$0 \to H_1(\hat{M}_\infty) \to H_1(\hat{M}_\infty, \hat{M}_0) \to \mathbb{Z}[t,t^{-1}] \to 0$$

which splits, because $\mathbb{Z}[t,t^{-1}]$ is free

$$\Rightarrow \quad H_1(\hat{M}_\infty, \hat{M}_0) \cong H_1(\hat{M}_\infty) \oplus \mathbb{Z}[t,t^{-1}].$$

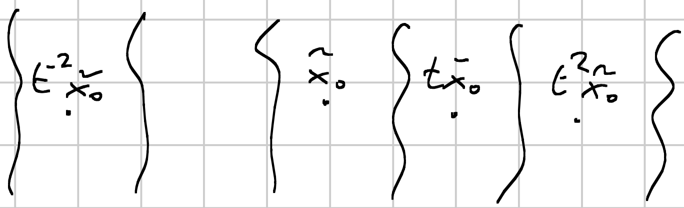
The statement on elementary ideals follows from the fact that, if A presents the

R- module M, then $\left(\frac{A}{0}\right)$ presents $M \oplus R$.

Hence, in order to compute the Alexander ideals of K, it is sufficient to compute the elementary ideals of $H_1(\tilde{M}_\infty, \tilde{M}_0)$.

Now we use cellular homology to compute $H_1(\tilde{M}_\infty, \tilde{M}_0)$. As a CW- complex, $\tilde{M}_\infty$ has a very visible structure.

$$0\text{-cells} = \tilde{M}_0 \implies C_0(\tilde{M}_0) \cong \mathbb{Z}[t,t^{-1}] \cdot \tilde{x}_0 \cong$$
$$\cong \mathbb{Z}[t,t^{-1}].$$



1- cells = liftings of the 1- cells $s_1, \dots s_n$ of M.

If $\tilde{s}_i$ denotes the lift of s_i starting at $\tilde{x}_0$

then $C_1(\tilde{M}_\infty) = \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}^n = (\mathbb{Z}[t,t^{-1}])^n$

with generators (over $\mathbb{Z}[t,t^{-1}]$) $\tilde{s}_1, \dots \tilde{s}_n$.

2-cells = liftings of 2-cells of M. Let $\tilde{r}_i$ be the 2-cell with initial vertex $\tilde{x}_0$ over the cell corresponding to r_i.

$$C_2(\tilde{M}_\infty) = \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}^k = (\mathbb{Z}[t,t^{-1}])^k$$

generated by $\tilde{r}_1, \dots, \tilde{r}_k$.

We know from cellular homology theory that $H_1(\tilde{M}_\infty, \tilde{M}_0)$ has generators $\tilde{s}_1, \dots, \tilde{s}_n$ (over $\mathbb{Z}[t,t^{-1}]$) (each $\tilde{s}_i$ is a cycle) and relations $\tilde{r}_1, \dots, \tilde{r}_k$.

If you are not comfortable with relative homology, observe $H_1(\tilde{M}_\infty, \tilde{M}_0) \cong H_1\left(\tilde{M}_\infty / \tilde{M}_0\right)$

cellular complex with "the same" 1- and 2-cells.

Next goal: Understand $\partial \tilde{r}_i$ as a linear combination of the $\tilde{s}_j$.

In the cellular chain complex of $\tilde{M}_\infty$

$$\partial \tilde{r}_i = \sum_{j=1}^{n} (\hat{D}_j r_i) \cdot \tilde{s}_j \ , \quad \hat{D}_j r_i \in \mathbb{Z}[t, t^{-1}].$$

How to compute $\hat{D}_j r_i$?

FOX DIFFERENTIAL CALCULUS.

G group. $\mathbb{Z}[G]$ is the group ring over G, i.e. the set of <u>finite</u> linear combinations

$$\sum_{g \in G} a_g \cdot g \ , \quad a_g \in \mathbb{Z}$$

with the obvious sum and product.

$$(1+g+\dots+g^m)(1-g) = 1-g^{m+1}$$
$$(1+h)(1+g) = 1+h+g+hg$$

Example: $\underset{\langle t \rangle}{\mathbb{Z}[\mathbb{Z}]} = \mathbb{Z}[t, t^{-1}]$

Let F_n be the free group on n generators.

Definition: $D : \mathbb{Z}[G] \to \mathbb{Z}[G]$ is a (Fox) derivative if:

① D is $\mathbb{Z}$-linear.

② $D(c_1 \cdot c_2) = D(c_1) \cdot \varepsilon(c_2) + c_1 \cdot D(c_2)$

where $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ $\varepsilon\left(\sum a_g g\right) = \sum a_g$.

Properties: ① $D(1) = D(1 \cdot 1) = D(1)\varepsilon(1) + 1 \cdot D(1) =$

$= 2D(1) \Rightarrow D(1) = 0$

$\Rightarrow D(m) = 0 \quad \forall m \in \mathbb{Z}$

② $0 = D(g \cdot g^{-1}) = D(g) \cdot \varepsilon(g^{-1}) + g\, D(g^{-1}) =$

$D(g) + g\, D(g^{-1}) \Rightarrow D(g^{-1}) = -g^{-1} D(g)$

③ $D(g^n) = (1 + g + \dots + g^{n-1})\, D(g)$

Proof: Easy induction.

④ $D(g^{-n}) = -(g^{-1} + g^{-2} + \dots + g^{-n})\, D(g)$.

If $G = F_n$, we have PARTIAL DERIVATIVES

$\frac{\partial}{\partial s_i} = D_i$, which are derivatives such that

$$\frac{\partial s_j}{\partial s_i} = \delta_{ij} \qquad \forall i, j$$

where $s_1, \dots s_n$ are the standard generators of F_n.

UNIQUENESS: obvious thanks to the fact $\frac{\partial}{\partial s_i}$ is prescribed on s_j, hence on s_j^{-1}, hence on all products of s_j, s_h^{-1} (i.e. on F_n), since $D(g_1 g_2)$ is determined by $D(g_1), D(g_2)$.

EXISTENCE: Every element $w \in F_n$ is a reduced word $w = s_{i_1}^{n_1} \cdot s_{i_2}^{n_2} \dots s_{i_\ell}^{n_\ell}$. Define

$$\frac{\partial}{\partial s_j}(w) = \left(\frac{\partial}{\partial s_j} s_{i_1}^{n_1}\right) + s_{i_1}^{n_1} \cdot \left(\frac{\partial}{\partial s_j} s_{i_2}^{n_2}\right) + \dots +$$

$$+ s_{i_1}^{n_1} \cdot \dots \cdot s_{i_{\ell-1}}^{n_{\ell-1}} \left(\frac{\partial}{\partial s_j} s_{i_\ell}^{n_\ell}\right)$$

extend by $\mathbb{Z}$-linearity, and check that it is a derivation.

In our situation, $G = \pi_1(M)$, $G = F_n / H$

$F_n = \langle s_1, \dots, s_n \rangle$, N normal closure of $r_1, \dots r_k$.

Let w be an element of F_n, $\overline{w}$ the corresponding element of G, and let

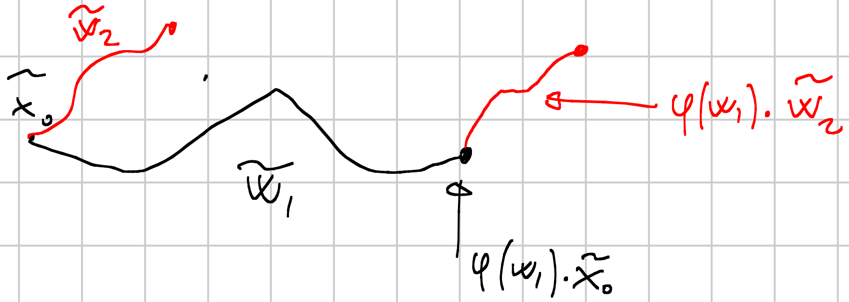
$$\varphi : F_n \to G \to H_1(M) = G/G' = \mathbb{Z}_n$$

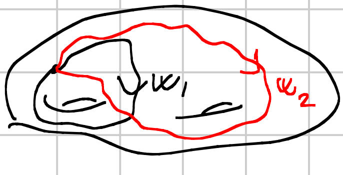
w is a word in the s_i, i.e. a path on the 1-skeleton of M. I am interested in $\tilde{w}$ (the lift of w starting at $\tilde{x}_0$), particularly when $w = r_i$ (in that case $\tilde{w} = \tilde{r}_i$ which the boundary of the 2-cell in $\tilde{M}_\infty$ corresponding to r_i and starting at $\tilde{x}_0$).

Properties of the map $w \longmapsto \tilde{w}$

$$F_n \longmapsto C_1(\tilde{M}, \tilde{M}_0)$$

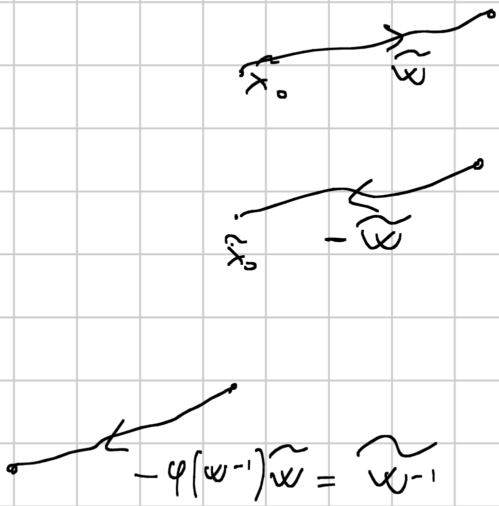
$$\widetilde{w_1 w_2} = \tilde{w}_1 + \varphi(w_1) \cdot \tilde{w}_2$$





Moreover, $\tilde{S}_i = \tilde{S}_i$ (notation is coherent)

($\widetilde{w^{-1}} = -\varphi(w^{-1})\,\tilde{w}$: check this geometrically).



Recall $\partial \tilde{r}_k = \sum_{j=1}^{n} (\hat{D}_j r_k)\cdot \tilde{s}_j$

I claim that $\hat{D}_j r_k = \varphi\left(\frac{\partial r_k}{\partial s_j}\right)$

$\varphi : F_n \to \mathbb{Z} = \langle t \rangle$ is extended by linearity to

$\varphi : \mathbb{Z}[F_n] \to \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$.

In fact, $\forall\, w_1, w_2 \in F_n$

$\widetilde{w_1 w_2} = \tilde{w}_1 + \varphi(w_1)\tilde{w}_2 \Rightarrow \hat{D}_j(w_1 w_2) = \hat{D}_j(w_1) + \varphi(w_1)\hat{D}_j(w_2)$

(in general, by $\hat{D}_j w$ I mean the coordinate w.r.t. $\tilde{s}_j$ of $\tilde{w}$).

Since $\tilde{s}_j = \tilde{s}_j$, $\hat{D}_j(s_i) = \delta_{ij}$

Hence, both $\hat{D}_j$ and $\varphi \circ \dfrac{\partial}{\partial s_j}$

satisfy $\quad s_i \longmapsto \delta_{ij}$

$$\hat{D}_j(w_1 w_2) = \hat{D}_j(w_1) + \varphi(w_1)\hat{D}_j(w_2)$$

$$\varphi\left(\frac{\partial}{\partial s_j} w_1 w_2\right) = \varphi\left(\frac{\partial}{\partial s_j} w_1\right) + \varphi(w_1)\,\varphi\left(\frac{\partial}{\partial s_j} w_2\right)$$

$\Rightarrow$ they coincide on the whole of F_n.

Theorem.: $H_1(\tilde{M}_\infty, \tilde{M}_0)$ admits a $n \times k$ presentation matrix A s.t.

$$A_{ij} = \varphi\left(\frac{\partial r_j}{\partial s_i}\right).$$