Suppose $A, A'$ are presentation matrices for the $R$-module $M$. Then

$$\begin{tikzcd} F \arrow[r, "A"] \arrow[d, shift right, "C"'] & E \arrow[r, "\pi"] \arrow[d, shift right, "B"'] \arrow[l, bend right, "D"'] & M \arrow[r] \arrow[d, equal] & 0 \\ F' \arrow[r, "A'"] \arrow[u, shift right, "C'"'] & E' \arrow[r] \arrow[u, shift right, "B'"'] & M \arrow[r] & 0 \end{tikzcd}$$

$$AD = B'B - Id$$
$$BA = A'C$$
$$B'A' = AC'$$

Using that $E, E', F, F'$ are free, we construct $B, B', C, C'$ s.t. the diagram commutes. $\pi \circ (B'B - Id) = 0 \Rightarrow \exists D$

We'd like to prove $A \sim A'$

First, up to adding and killing generators, we may suppose $A, A'$ with the same number of rows.

(it was described by $A \sim \left(\begin{array}{c|c} A & 0 \\ \hline 0 & Id \end{array}\right)$)

$$A \sim \left(\begin{array}{c|c} A & 0 \\ \hline 0 & Id \end{array}\right) \sim \left(\begin{array}{c|c} A & B' \\ \hline 0 & Id \end{array}\right) \sim \left(\begin{array}{c|c|c} A & B' & 0 \\ \hline 0 & Id & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{c|c|c} A & B' & B'A' \\ \hline 0 & Id & A' \end{array}\right) = \left(\begin{array}{c|c|c} A & B' & AC' \\ \hline 0 & Id & A' \end{array}\right) \sim$$

$$\sim \left(\begin{array}{c|c|c} A & B' & 0 \\ \hline 0 & Id & A' \end{array}\right) \sim \left(\begin{array}{c|c|c|c} A & B' & 0 & B'B \\ \hline 0 & Id & A' & B \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c} A & B' & 0 & AD + \mathrm{Id} \\ \hline 0 & \mathrm{Id} & A' & B \end{array}\right) \sim \left(\begin{array}{c|c|c|c} A & B' & 0 & \mathrm{Id} \\ \hline 0 & \mathrm{Id} & A' & B \end{array}\right)$$

$$A' \sim \left(\begin{array}{c|c|c|c} A' & B & 0 & \mathrm{Id} \\ \hline 0 & \mathrm{Id} & A & B' \end{array}\right)$$

obtain one from the other via permutation of rows and columns

This proves that $A \sim A'$.

Definition: $M$ finitely presented $R$-module with presentation matrix $A$ with $m$ rows (which means $m$ generators for $M$). The **$r$-th elementary ideal $E_r(M)$** is the ideal in $R$ generated by $(m-r+1) \times (m-r+1)$ minors of $A$.

It is well-defined thanks to the theorem stating that all presentation matrices are equivalent (the only non-trivial fact being that $A$ and $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$ generate the same ideals; use laplace). By laplace, a $(d+1)\times(d+1)$ minor is generated by $d\times d$ minors. $\Longrightarrow E_r(M) \subseteq E_{r+1}(M)\ \forall r$.

We agree that $E_r(M) = R$ if $r > m$

$E_r(M) = 0$ if $r \leq 0$.

Exercises: ① $\mathbb{Z}[t,t^{-1}]$ is not a P.I.D., but it is a U.F.D. so $E_r(M)$ is not principal in general, but it has a G.C.D. (greatest common divisor), which is named the $r$-th Alexander polynomial of $M$ (of a $L$, when $M = H_1(\tilde{E}_\infty)$ $E = S^3 \setminus L$).

② The invertibles of $\mathbb{Z}[t,t^{-1}]$ are $\pm t^n$, $n \in \mathbb{Z}$. Hence Alexander polynomials are well-defined only up to multiplication by $\pm t^n$

Definition: "The" Alexander polynomial is the first Alexander polynomial.

Notation: $f, g \in \mathbb{Z}[t, t^{-1}]$. We write

$$f \doteq g \iff f = \pm t^n g.$$

Example: $K$ the unknot. $S^3 \setminus K \cong$
$\cong \mathbb{R}^3 \setminus \{\text{straight line}\} \cong S^1 \times \mathbb{R}^+ \times \mathbb{R}$,
hence $\tilde{E}_\infty = \tilde{E}$ (universal covering) $= \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$
$\cong \mathbb{R}^3 \Rightarrow H_1(\tilde{E}_\infty) = 0$, which is presented
e.g. by the matrix $(1)$. Hence $E_1(K) = R$
and the first Alexander polynomial is

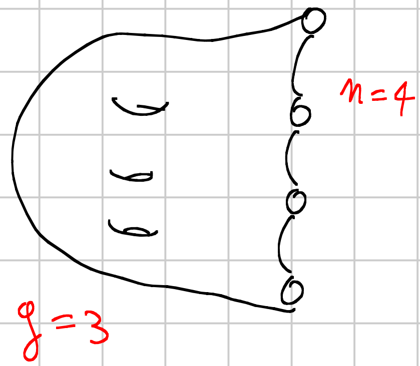
$$\Delta_K(t) = 1.$$

notation for the 1-st Alexander polynomial

Problem: How to compute $\Delta_L$ in general?

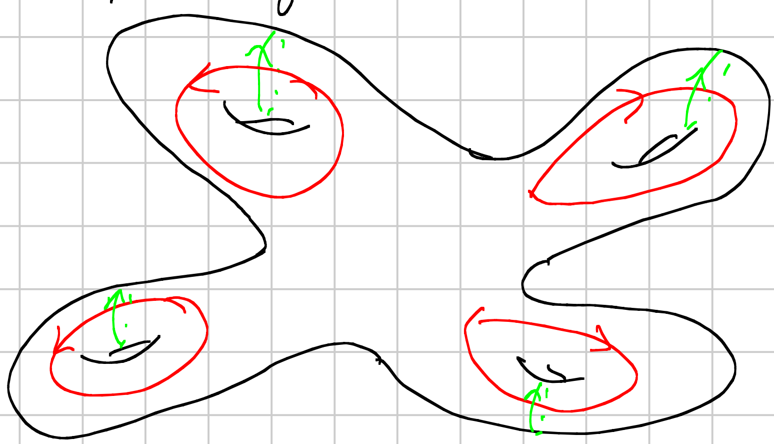
# 1-st Homology of surfaces

Orientable surface is $\Sigma_{g,n} =$

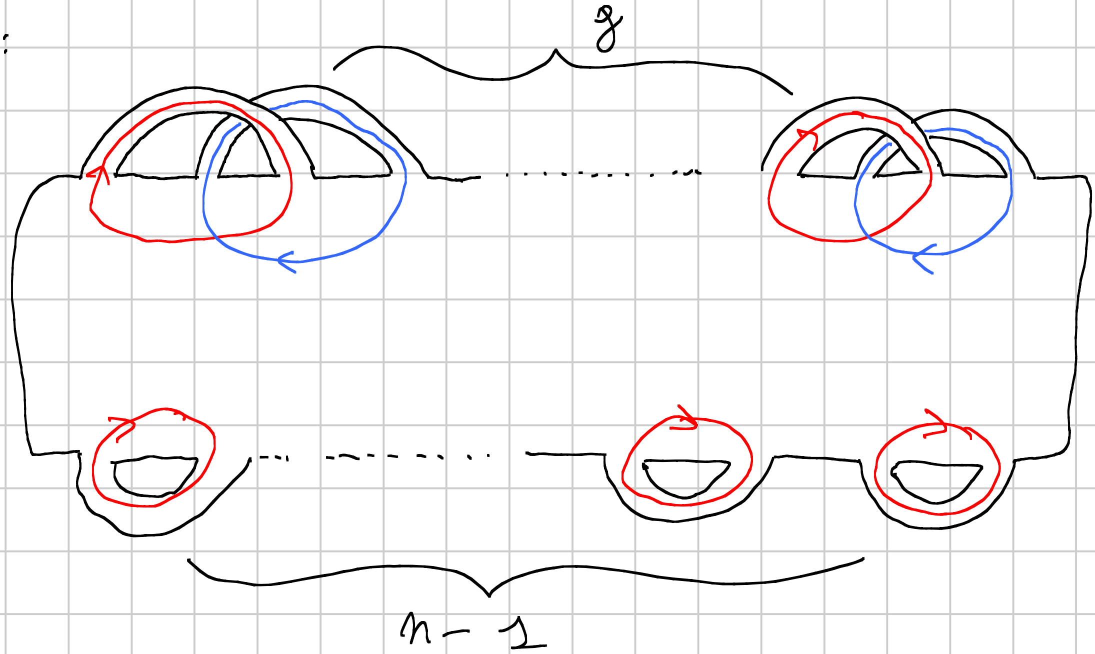



If $n=0$, $\quad H_1(\Sigma_{g,0}) = H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$

with the following basis



If $n>0$ (i.e. $\partial\Sigma_{g,n} \neq \emptyset$) the $\Sigma_{g,n}$ is as follows:



$H_1(\Sigma_{g,n}) \cong \mathbb{Z}^{2g+n-1}$ with the basis above

We come back to our setting: $L \subseteq S^3$ and we look for a presentation of $H_1(\tilde{E}_\infty)$, $E = S^3 \setminus L$.

We also fix a Seifert surface $S$ for $L$.

$n$ = # components of $L$, $g$ = genus$(S)$.

Prop.: $H_1(S) = \mathbb{Z}^{2g+n-1} \cong H_1(S^3 \setminus S)$

and $\exists$ a bilinear $\beta: H_1(S^3 \setminus S) \times H_1(S) \to \mathbb{Z}$

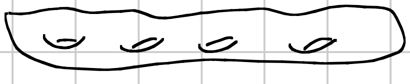
s.t.

$$\beta([c],[d]) = lk(c,d)$$

$\forall$ simple closed oriented loops $c \subseteq S^3 \setminus S$, $d \subseteq S$

Proof: $V = \overline{N(S)}$, $V' = \overline{S^3 \setminus V}$

By representing $S$ as a disk with (knotted) bands, we understand $V$ = handlebody of genus $2g+n-1$ (handlebody of genus $g$ =

 as a 3-manifold with boundary).

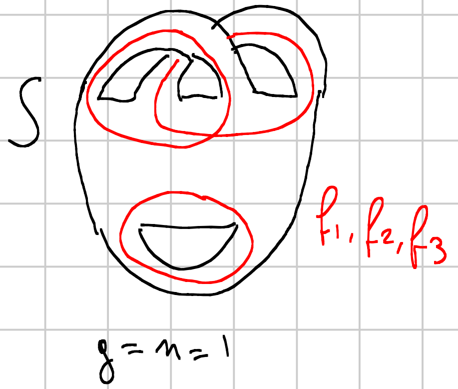
Of course, $V$ is a knotted handlebody, with one (knotted) solid cylinder coming from each (knotted) band of $S$. On $\partial V = \Sigma_{2g+n-1}$ we take the standard basis for $H_1$ which can be constructed as follows:

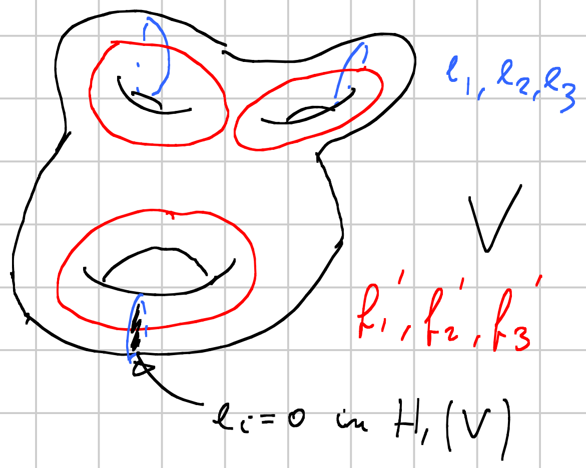
$f_1, \dots, f_{2g+n-1}$ "standard" basis for $H_1(S)$

$f_1', \dots, f_{2g+n-1}' \in H_1(\partial V)$ s.t. $f_i' = f_i$ in $H_1(V)$

$e_1, \dots, e_{2g+n-1} \in H_1(\partial V)$ s.t. $e_i$ bounds a disk in $V$

"dual" to $f_i$




From Meyer-Vietoris

$$\begin{array}{ccccccc} & & V \cap V' & & & & \\ & & \shortparallel & & & & \\ H_2(S^3) & \to & H_1(\partial V) & \xrightarrow{\varphi} & H_1(V) \oplus H_1(V') & \to & H_1(S^3) \\ \shortparallel & & & & & & \shortparallel \\ 0 & & f_i' & \longmapsto & (f_i, \; ?) & & 0 \\ & & e_i & \longmapsto & (0, \; ?) & & \end{array}$$

$\varphi$ is an isomorphism $\Longrightarrow$ the $e_i$ define a basis of $H_1(V') \cong H_1(S^3 \setminus S)$.

Now define an orientation on $f_i, e_j$ s.t. $lk(f_i, e_i) = +1$ (recall that $e_i$ bounds a disc intersecting $f_i$ transversely once).

(This is not bothered by knottedness of $V, V'$).

Since this disc is disjoint from $f_j$, $j \neq i$, $lk(f_j, e_i) = \delta_{ij}$. Now define

$$\beta\left(\sum a_i [e_i], \sum b_j [f_j]\right) = \sum a_i b_i$$

I now check that $\beta([c], [d]) = lk(c, d)$.

$\forall i$, $lk(c, f_i) = [c]$ in $H_1(S^3 \setminus f_i) =$

$= \left[\sum a_k [e_k]\right] = \sum a_k \, lk(e_k, f_i) = a_i$

(I am assuming $[c] = \sum a_k [e_k]$, $[d] = \sum b_k [f_k]$).

Thus $lk(c, d) = [d]$ in $H_1(S^3 \setminus c) =$

$= \left[\sum b_k [f_k]\right] = \sum b_k \, lk(f_k, c) = \sum b_k a_k =$

$= \beta([c], [d])$.

Facts: From elementary linear algebra (over $\mathbb{Z}$), for $\boxed{\text{every}}$ basis $f_1, \dots, f_{2g+n-1}$ of $H_1(S)$ there exists a dual basis $e_1, \dots, e_{2g+n-1}$ s.t. $\beta(f_i, e_j) = \delta_{ij}$.

Remember $S$ is oriented $\Longrightarrow N(S) \cong S \times [-1,1]$ hence we have embeddings $i^+ : S \to S^3$ $i^- : S \to S^3$ $i^{\pm}(x) = (x, \pm 1)$. We define

$$\alpha : H_1(S) \times H_1(S) \to \mathbb{Z}$$

$$\alpha(x, y) = \beta(x^-, y) = \beta(y^+, x)$$

where $x^{\pm} = i^{\pm}_*(x)$, $y^{\pm} = i^{\pm}_*(y)$.

clear from a geometric point of view

$$\beta : H_1(S^3 \setminus S) \times H_1(S) \to \mathbb{Z}$$

Remark: Fix dual basis $f_i$, $e_i$ of $H_1(S)$, $H_1(S^3 \setminus S)$. Let $A$ be the matrix representing $\alpha$ w.r.t. the $f_i$, i.e.

$A_{ij} = \alpha(f_i, f_j) = \beta(f_i^-, f_j)$.

Then $f_i^- = \sum_j A_{ij} e_j$ , $f_i^+ = \sum_j A_{ji} e_j$

In fact, $f_i^-$ and $\sum_j A_{ij} e_j$ have the same pairings with each $f_k$ via $\beta$, hence they are equal. The same for $f_i^+$.

Theorem: $A$ representing $\alpha$ w.r.t. a basis of $H_1(S)$. Then $tA - {}^tA$ is a presentation matrix for $H_1(\tilde{E}_\infty)$.