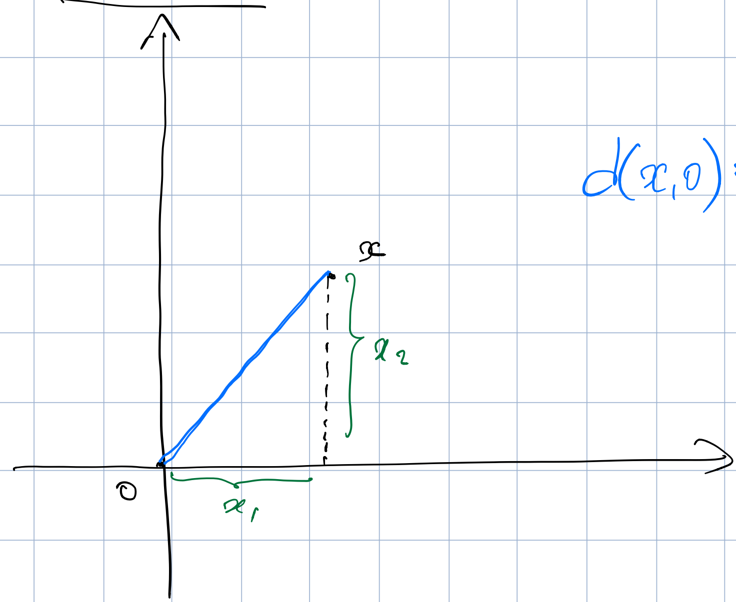
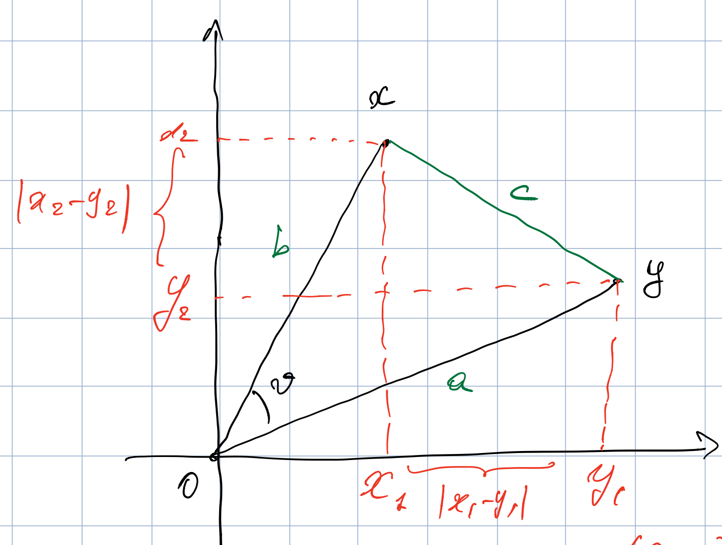
Geometrie



$$d(x,0) = \sqrt{x_1^2 + x_2^2}$$



$$c^2 = a^2 + b^2 - 2ab\cos(\vartheta)$$

$$\cos(\vartheta) = \frac{a^2 + b^2 - c^2}{2ab}$$

$$c = d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$\cos(\vartheta) = \frac{(x_1^2 + x_2^2) + (y_1^2 + y_2^2) - \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)}{2\sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}}$$

$$= \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}}.$$

$$d(0,x)^2 = x_1 \cdot x_1 + x_2 \cdot x_2$$

$$\cos(\vartheta(x,y)) = \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}}$$

Def: dati $x, y \in \mathbb{R}^2$ chiamo loro prodotto scalare (standard di $\mathbb{R}^2$)

$$\langle x | y \rangle_{\mathbb{R}^2} = x_1 y_1 + x_2 y_2 = (x_1, x_2) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = {}^t x \cdot y .$$

Inoltre dato $x \in \mathbb{R}^2$ chiamo norma di x

$$\|x\|_{\mathbb{R}^2} = \sqrt{\langle x | x \rangle_{\mathbb{R}^2}} \geq 0 .$$

Ho:

$$d(0,x) = \|x\|_{\mathbb{R}^2}$$

$$d(x,y) = \|x - y\|_{\mathbb{R}^2}$$

$$\cos(\vartheta(x,y)) = \frac{\langle x | y \rangle_{\mathbb{R}^2}}{\|x\|_{\mathbb{R}^2} \cdot \|y\|_{\mathbb{R}^2}} .$$

Uguale in $\mathbb{R}^n$:

prod. scal. standard : $\langle \cdot | \cdot \rangle_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(x, y) \longmapsto {}^t x \cdot y$$

norma associata : $\|x\|_{\mathbb{R}^n} = \sqrt{\langle x | x \rangle_{\mathbb{R}^n}} \geq 0 .$

Def: dato V spazio vettoriale su $\mathbb{R}$ e

$$\langle \cdot | \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

(funzione con argomento coppia di vettori e valore numero)

dico che essa è:

- **bilineare** se è lineare in ciascun argomento fissato l'altro, cioè:

$$\langle \alpha x + \beta z \mid y \rangle = \alpha \langle x \mid y \rangle + \beta \langle z \mid y \rangle \quad \text{sinistra}$$

$$\langle x \mid \alpha y + \beta z \rangle = \alpha \langle x \mid y \rangle + \beta \langle x \mid z \rangle \quad \text{destra}$$

- **simmetrica** se $\langle y \mid x \rangle = \langle x \mid y \rangle$

- **definita positiva** se $\langle x \mid x \rangle > 0 \quad \forall x \neq 0$

- **prodotto scalare** se valgono le 3 precedenti.

Oss: $\langle \cdot \mid \cdot \rangle_{\mathbb{R}^n}$ è un prodotto scalare:

1) $\langle \alpha x + \beta z \mid y \rangle_{\mathbb{R}^n} = {}^t(\alpha x + \beta z) \cdot y = (\alpha \, {}^t x + \beta \, {}^t z) \cdot y$

$= \alpha \, {}^t x \cdot y + \beta \, {}^t z \cdot y = \alpha \langle x \mid y \rangle_{\mathbb{R}^n} + \beta \langle z \mid y \rangle_{\mathbb{R}^n}$

(a dx analogo)

2) $\langle \underset{n\times 1}{y} \mid \underset{n\times 1}{x} \rangle_{\mathbb{R}^n} = \underbrace{\underset{1\times n}{{}^t y} \cdot \underset{n\times 1}{x}}_{\substack{1\times 1 \\ \text{numero}}} = {}^t({}^t y \cdot x) = {}^t x \cdot y = \langle x \mid y \rangle_{\mathbb{R}^n}$

(In generale: ${}^t\underbrace{(\underbrace{\underset{p\times q}{M} \cdot \underset{q\times r}{N}}_{p\times r})}_{r\times p} = \underbrace{\underset{r\times q}{{}^t N} \cdot \underset{q\times p}{{}^t M}}_{r\times p}$

$$\left({}^t(M\cdot N)\right)_{ij} = (M\cdot N)_{ji} = \sum_{k=1}^{n} (M)_{jk}\cdot (N)_{ki}$$

$$= \sum_{k=1}^{n} ({}^tN)_{ik} \cdot ({}^tM)_{kj} = ({}^tN\cdot {}^tM)_{ij}.\Big)$$

3) def. pos: $\langle x|x\rangle_{\mathbb{R}^n} = {}^tx\cdot x = (x_1 \dots x_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$= x_1^2 + \dots + x_n^2 > 0 \quad \text{se } x \neq 0.$$

Esempi: 1) $V = \mathcal{M}_{m\times n}(\mathbb{R})$; $\langle A|B\rangle = \mathrm{tr}(\underbrace{{}^tA\cdot B}_{\substack{n\times m \;\; m\times n \\ n\times n}})$

Affermo che è prod. scal:

- bilineare

sin: $\langle \lambda A + \mu C | B\rangle = \mathrm{tr}\left({}^t(\lambda A + \mu C)\cdot B\right)$
$= \mathrm{tr}\left((\lambda\, {}^tA + \mu\, {}^tC)\cdot B\right)$
$= \mathrm{tr}(\lambda\cdot {}^tA\cdot B + \mu\cdot {}^tC\cdot B)$
$= \lambda\cdot \mathrm{tr}({}^tA\cdot B) + \mu\cdot \mathrm{tr}({}^tC\cdot B)$
$= \lambda\langle A|B\rangle + \mu\langle C|B\rangle.$

dx: analogo.

Anzi: se una funz. di due argomenti è lin. a sin. e simmetrica allora è lin. a dx: facile.

- simmetrica: $\langle B|A\rangle = \mathrm{tr}({}^tB\cdot A) = \mathrm{tr}\left({}^t({}^tB\cdot A)\right)$
$= \mathrm{tr}({}^tA\cdot B) = \langle A\cdot B\rangle.$

- definita positiva : $\langle A|A\rangle = tr({}^tA\cdot A) = \sum_{i=1}^{m} ({}^tA\cdot A)_{ii}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} ({}^tA)_{ij} \cdot (A)_{ji}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \left((A)_{ji}\right)^2 =$$ somma dei quadrati di tutti i coeff di A

> 0 per $A \neq 0$.

2) $V = C^0([0,1], \mathbb{R})$ (oppure $V = \mathbb{R}[t]$)

$$\langle x|y\rangle = \int_0^1 x(t)\cdot y(t)\, dt$$

- bilineare : dx : $\langle x|\alpha y + \beta z\rangle = \int_0^1 x(t)\cdot(\alpha y + \beta z)(t)\, dt$

$$= \int_0^1 x(t)\cdot\left(\alpha\cdot y(t) + \beta\cdot z(t)\right) dt$$

$$= \int_0^1 \left(\alpha\cdot x(t)\cdot y(t) + \beta\cdot x(t)\cdot z(t)\right) dt$$

$$= \alpha\cdot \int_0^1 x(t)\cdot y(t)\, dt + \beta \int_0^1 x(t)\cdot z(t)\, dt$$

$$= \alpha\cdot\langle x|y\rangle + \beta\cdot\langle x|z\rangle$$

- simmetrica : $\langle y|x\rangle = \int_0^1 y(t)\cdot x(t)\, dt = \int_0^1 x(t)\cdot y(t)\, dt = \langle x|y\rangle$

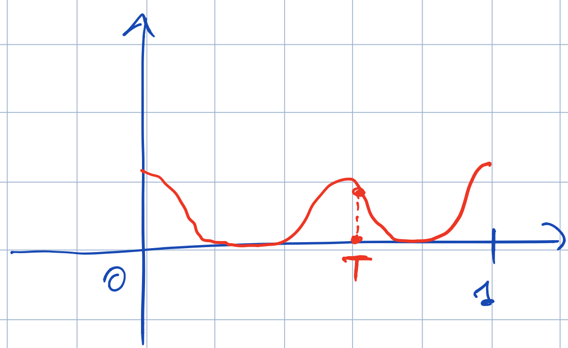
- def. pos : devo provare che se $x \neq 0$ ho $\langle x|x\rangle > 0$.

$x \neq 0$ significa che $\exists\, T \in [0,1]$ t.c. $x(T) \neq 0$.

$$\langle x | x \rangle = \int_0^1 x(t) \cdot x(t)\, dt = \int_0^1 x(t)^2\, dt$$

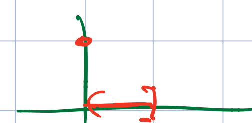
$x(t)^2 \geq 0 \quad \forall t$

$x(T)^2 > 0$

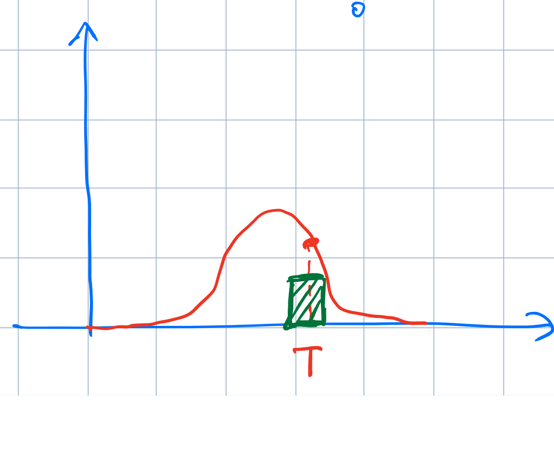


ATT: se ho $y : [0,1] \to \mathbb{R}$ integrabile con $y(t) \geq 0 \ \forall t$ e $y(T) > 0$ posso avere $\int_0^1 y(t)\, dt = 0$; ad es:

$$y(t) = \begin{cases} 1 & \text{per } t = 0 \\ 0 & \text{per } t \in (0,1] \end{cases}$$



Però se y è continua ho $\int_0^1 y(t)\, dt > 0$



Def: V sp. vett. su $\mathbb{R}$; prod scal è $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{R}$
- bilineare
- simmetrica
- def. pos.

Oss: per la bilinearità $\langle 0|v\rangle = 0 \quad \forall v$
$\langle v|0\rangle = 0 \quad \forall v$

$\Rightarrow \langle 0|0\rangle = 0$.

Q: quali sono i prod. scal. su $\mathbb{R}^n$?

Q1: quali sono le funz. bilineari su $\mathbb{R}^n$ } subito

Q2: — — — — — — simmetriche " } subito

Q3: — — — — — — — — def. pos " } poi

$\boxed{A1}$ Def: data $A \in M_{n\times n}(\mathbb{R})$ pongo

$$\langle\cdot|\cdot\rangle_A : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\langle x|y\rangle_A = {}^t x \cdot A \cdot y.$$

Oss: $\langle\cdot|\cdot\rangle_A$ è sempre bilineare:

sin: $$\langle \alpha x + \beta z|y\rangle_A = {}^t(\alpha x + \beta z) \cdot A \cdot y = (\alpha\, {}^t x + \beta\, {}^t z) \cdot A \cdot y$$
$$= \alpha \cdot {}^t x \cdot A \cdot y + \beta \cdot {}^t z \cdot A \cdot y$$
$$= \alpha \langle x|y\rangle_A + \beta \langle z|y\rangle_A$$

dx: analogo.

Prop: le appl. bil. su $\mathbb{R}^n$ sono tutte e sole quelle del tipo $\langle\cdot|\cdot\rangle_A$ con $A \in M_{n\times n}(\mathbb{R})$.

Dimo: visto che ogni $\langle\cdot|\cdot\rangle_A$ è bil.

Data $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ cerchiamo A t.c. $f = \langle\cdot|\cdot\rangle_A$.

Se tale A esiste ho in part. che

$$f(e_i, e_j) = \langle e_i | e_j \rangle_A = {}^t e_i \cdot A \cdot e_j$$

$$= \begin{pmatrix} 0 & \cdots & \underset{i}{1} & \cdots & 0 \end{pmatrix} \begin{array}{c} \\ i \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \\ j \end{array} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} \\ j \\ \\ \end{array}$$

$$= a_{ij}.$$

Ora data f pongo $a_{ij} = f(e_i, e_j)$, $A = (a_{ij})_{i,j=1\dots n}$ e verifico che $f = \langle\cdot|\cdot\rangle_A$:

$$f, \langle\cdot|\cdot\rangle_A \ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$f(x,y) = f\left(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j\right)$$

$$= \sum_{i=1}^n x_i \, f\left(e_i, \sum_{j=1}^n y_j e_j\right)$$

$$= \sum_{i=1}^n x_i \sum_{j=1}^n y_j \cdot \underbrace{f(e_i, e_j)}_{a_{ij}}$$

$$= \sum_{i=1}^{n} \underbrace{x_i}_{({}^t x)_i} \cdot \underbrace{\sum_{j=1}^{n} a_{ij}\, y_j}_{(A\cdot y)_i}$$

$$\underbrace{\qquad\qquad\qquad}_{{}^t x \cdot A \cdot y \;=\; \langle x | y \rangle_A .}$$

$\boxed{A2}$ **Prop:** $\langle \cdot | \cdot \rangle_A$ è simm $\Longleftrightarrow$ A è simm.

Dimo: $\Longleftarrow$: Supponiamo A simm, cioè ${}^t A = A$.

Allora $\langle y | x \rangle_A = {}^t y \cdot A \cdot x = {}^t({}^t y \cdot A \cdot x)$

$= {}^t x \cdot {}^t A \cdot y = {}^t x \cdot A \cdot y = \langle x | y \rangle_A .$

$\Longrightarrow$: Supponiamo $\langle \cdot | \cdot \rangle_A$ simm, cioè
$\langle y | x \rangle_A = \langle x | y \rangle_A \quad \forall x, y$

In part: $\underbrace{\langle e_i | e_j \rangle_A}_{a_{ij}} = \underbrace{\langle e_j | e_i \rangle_A}_{a_{ji}}$

Oss: se $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ diagonale

$$\langle x | x \rangle_A = (x_1 \cdots x_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= (x_1 \cdots x_n) \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{pmatrix} = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$$

è def. pos. $\Longleftrightarrow$ $\lambda_1, \dots, \lambda_n > 0$.

Oss: se $A = {}^tB \cdot B$ * con B invertibile ho

$$\langle x|x\rangle_A = {}^tx \cdot A \cdot x = {}^tx \cdot {}^tB \cdot B \cdot x$$
$$= {}^t(B \cdot x) \cdot (B \cdot x) = \|B \cdot x\|^2_{\mathbb{R}^n}$$

Sempre ≥ 0; nulla solo se $B \cdot x = 0$, ma B è invertibile $\Longrightarrow$ nulla solo se $x = 0$.

Dunque: $\langle \cdot | \cdot \rangle_A$ è un prod scal.

* $A = {}^tB \cdot B \Rightarrow {}^tA = {}^t({}^tB \cdot B) = {}^tB \cdot B = A$
cioè A è simm.

Torniamo a V sp. vett. su $\mathbb{R}$ con
$\langle \cdot | \cdot \rangle : V \times V \to \mathbb{R}$ prod. scal. (bil, simm, def. pos.)

Def: chiamo norma associata a $\langle \cdot | \cdot \rangle$ la

$$\|\cdot\| : V \longrightarrow \mathbb{R}$$
$$\|v\| = \sqrt{\langle v|v\rangle}$$

(la radice ≥ 0) c'è poiché $\langle v|v\rangle \geq 0$

Oss: conoscendo $\|\cdot\|$ conosco anche $\langle \cdot | \cdot \rangle$.

Analogo al fatto che nel piano, se conosco la distanza conosco gli angoli. (Falso su $\mathbb{C}$.)

$$\|v+w\|^2 = \langle v+w | v+w \rangle \underset{\text{sim}}{=} \langle v | v+w \rangle + \langle w | v+w \rangle$$

$$\underset{\text{dx}\times 2}{=} \underbrace{\langle v|v\rangle}_{\|v\|^2} + \underbrace{\langle v|w\rangle + \langle w|v\rangle}_{2\langle v|w\rangle} + \underbrace{\langle w|w\rangle}_{\|w\|^2}$$

$$\Longrightarrow \quad \langle v|w\rangle = \frac{1}{2}\left(\|v+w\|^2 - \|v\|^2 - \|w\|^2\right)$$

Propietà di $\|\cdot\|$:

- $\|v\| > 0$ per $v \neq 0$, $\|0\| = 0$
- $\|\lambda \cdot v\| = \sqrt{\langle \lambda v | \lambda v\rangle} = \sqrt{\lambda^2 \cdot \langle v|v\rangle} = |\lambda| \cdot \|v\|$

Prop (disuguaglianza di Cauchy-Schwartz) :

$$|\langle v|w\rangle| \leq \|v\| \cdot \|w\|$$

e l'uguaglianza vale solo se sono multipli tra loro.

Dim : multipli : $w = \lambda \cdot v \Rightarrow$

$$|\langle v|w\rangle| = |\langle v|\lambda v\rangle| = |\lambda \cdot \langle v|v\rangle| = |\lambda| \cdot \|v\|^2$$

$$\|v\| \cdot \|w\| = \|v\| \cdot \|\lambda \cdot v\| = |\lambda| \cdot \|v\|^2 \qquad \text{OK}$$

$v = \lambda \cdot w$ analogo.

non multipli : allora ho $tv + w \neq 0 \quad \forall t \in \mathbb{R}$

διότι $\quad \| tv+w \|^2 > 0 \quad \forall t \in \mathbb{R}$

$$\Rightarrow \langle tv+w \mid tv+w \rangle = t^2 \cdot \|v\|^2 + 2t \cdot \langle v \mid w \rangle + \|w\|^2 > 0 \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \Delta/4 < 0 \Rightarrow \langle v \mid w \rangle^2 - \|v\|^2 \cdot \|w\|^2 < 0$$

$$\Rightarrow \langle v \mid w \rangle^2 \leq \|v\|^2 \cdot \|w\|^2$$

$$\Rightarrow |\langle v \mid w \rangle| \leq \|v\| \cdot \|w\|. \qquad \blacksquare$$