# Esercitazione 17/11/16

Per lo spazio vettoriale $V$ assegnato, e sottospazi $W, Z \subset V$, provare che si ha la decomposizione per somma diretta $V = W \oplus Z$, ed esibire le proiezioni associate $p$ e $q$, verificare $p \circ p = p$, $q \circ q = q$, $p \circ q = q \circ p = 0$.

5.2.4. $V = \mathbb{R}^2$, $W = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$, $Z = \mathrm{Span}\left\{\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right\}$

Poiché $\dim(V) = 2 = \dim W + \dim Z$, per verificare che

$V = W \oplus Z$, basta verificare che $V = W + Z$

Vero poiché $W + Z = \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix}}_{\text{linearmente indip.}} \right\} = \mathbb{R}^2 = V$.

$V = W \oplus Z$, $v \in V$ $\quad v = \underset{W}{\underset{\Uparrow}{w}} + \underset{Z}{\underset{\Uparrow}{z}}$ $\rightarrow$ scrittura è unica

$p$ proiezione su $W$ $\quad p(v) = w$, $q$ proiez. su $Z$ $\quad q(v) = z$.

Scegliamo come base per $V$ $\quad B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\}$

base per $W$ $\quad$ base per $Z$

$$[P]_B^B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\in W} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\in Z}$$

$$P \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix}-3\\2\end{pmatrix} = 0\cdot\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}-3\\2\end{pmatrix} \qquad \begin{pmatrix}2\\1\end{pmatrix} = 1\cdot\begin{pmatrix}2\\1\end{pmatrix} + 0\cdot\begin{pmatrix}-3\\2\end{pmatrix}$$

$$p\begin{pmatrix}-3\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad\rightsquigarrow\quad \begin{pmatrix}-3\\2\end{pmatrix} = \underset{\substack{\cap\\ W}}{\begin{pmatrix}0\\0\end{pmatrix}} + \underset{\substack{\cap\\ Z}}{\begin{pmatrix}-3\\2\end{pmatrix}}$$

$$[q]^{B}_{B} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$p \circ p = p \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$q \circ q = q \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$p \circ q = 0 \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$p = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] \qquad q = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_n \end{array}\right]$$

5.2.5. $V = \mathbb{R}^2$. $W = \mathrm{Span}\left\{\begin{pmatrix}3\\-1\end{pmatrix}\right\}$ $Z = \{x \in \mathbb{R}^2 : 5x_1 + 2x_2 = 0\}$

- $W + Z = \mathbb{R}^2$ $Z = \mathrm{Span}\left\{\begin{pmatrix}2\\-5\end{pmatrix}\right\}$

equivalenti

$W + Z = \mathrm{Span}\left\{\begin{pmatrix}3\\-1\end{pmatrix}, \begin{pmatrix}2\\-5\end{pmatrix}\right\} = \mathbb{R}^2$. $\square$

- $W \cap Z = \{0\}$ $\longrightarrow$ per verificare questo e' sufficiente verificare che $\begin{pmatrix}3\\-1\end{pmatrix} \notin Z$

5.3.1. Provare che, $\forall A \in M_{m \times n}(\mathbb{R})$ vale $A \cdot I_n = A$

$I_m \cdot A = A$

$$(A \cdot I_n)_{ij} = \sum_{h=1}^{n} (A)_{ih} \cdot (I)_{hj} = (A)_{ij} \cdot (I)_{jj} = (A)_{ij} \rightarrow \forall i = 1, \dots, m$$

$\forall j = 1, \dots, n$

$$\overset{\shortparallel}{\left((A)_{i1} \cdot (I)_{1j}\right)} + \left((A)_{i2} \cdot (I)_{2j}\right) + \dots + \left((A)_{in} \cdot (I)_{nj}\right)$$

Es. 5.3.4. Date $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, $B \in \mathcal{M}_{k\times h}$, verificare che la formula $f(X) = A \cdot X \cdot B$ definisce un'applicazione lineare $f: \mathcal{M}_{n\times k}(\mathbb{R}) \to \mathcal{M}_{m\times h}(\mathbb{R})$ e che $\operatorname{Ker}(f) = \{X : X(\operatorname{Im}(B)) \subset \operatorname{Ker} A\}$.

$X \in \mathcal{M}_{n\times k}$ $\quad A \cdot X \cdot B = \underbrace{A}_{m\times n} \cdot \underbrace{(X \cdot B)}_{n\times h} \to \in \mathcal{M}_{m\times h}$

- Verifichiamo che $f(X+Y) = f(X) + f(Y)$ $\forall X, Y \in \mathcal{M}_{n\times k}(\mathbb{R})$

$$f(X+Y) = A\cdot(X+Y)\cdot B = A\cdot(X\cdot B+Y\cdot B) = \underbrace{A\cdot X\cdot B}_{= f(X)} + \underbrace{A\cdot Y\cdot B}_{= f(Y)}$$

distributività di + rispetto a $\cdot$

- $f(\lambda X) = \lambda\cdot f(X)$ (esercizio)

- $X \in \mathrm{Ker}(f) \iff A\cdot X\cdot B = 0 \iff \forall v\in\mathbb{R}^h\ \underbrace{(A\cdot X\cdot B)}_{m\times h}\cdot v = \overset{\mathbb{R}^m}{0}$ ...

$\iff \forall v\in\mathbb{R}^h\ X\cdot \underbrace{B\cdot v}_{\mathrm{Im}(B)} \in \mathrm{Ker}(A) \iff$ $\qquad$ $= A\cdot(X\cdot B\cdot v)$

$\Leftrightarrow \forall w \in \operatorname{Im}(B)\ X_i w \in \operatorname{Ker}(A) \Longleftrightarrow X(\operatorname{Im}(B)) \subset \operatorname{Ker}(A)$ $\square$

$A \in \mathcal{M}_{m \times n}(\mathbb{R}) \rightsquigarrow f_A : \mathbb{R}^n \to \mathbb{R}^m$ $\quad f_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$\operatorname{Im}(A) = \{A \cdot x \mid x \in \mathbb{R}^n\}$
$\cap$
$\mathbb{R}^m$

$A = \left[ c_1 \mid c_2 \mid \dots \mid c_n \right]$ $\quad c_i \in \mathbb{R}^m$

$\operatorname{Im}(A) = \operatorname{Span}\{c_1, \dots, c_n\}$

$\mathrm{Span}\{c_1, \dots, c_n\} \subseteq \mathrm{Im}(A) \qquad c_i \in \mathrm{Im}(A)$

$\mathrm{Im}(A) \subset \mathrm{Span}\{c_1, \dots, c_n\}$

$v \in \mathbb{R}^n \quad v = \lambda_1 e_1 + \dots + \lambda_n e_n \quad A \cdot v = A(\lambda_1 e_1 + \dots + \lambda_n e_n) =$

$= \lambda_1 \cdot A(e_1) + \dots + \lambda_n \cdot A(e_n) = \lambda_1 \cdot c_1 + \dots + \lambda_n c_n$