Esercitazione 2/11/16

4.4.2. Estrarre dai vettori $v_1, \dots, v_h$ assegnati una base dello spazio V dato.

a) $V = \mathbb{R}^3$, $\underbrace{v_1 = \begin{pmatrix} 4 \\ 6 \\ -10 \end{pmatrix}}_{\text{SI}}$, $v_2 = \cancel{\begin{pmatrix} -6 \\ -9 \\ 15 \end{pmatrix}}$, $\underbrace{v_3 = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}}_{\text{SI}}$, $v_4 = \cancel{\begin{pmatrix} -1 \\ 13 \\ -12 \end{pmatrix}}$ $\Downarrow$ $v_4 = v_1 - v_3$, $\underbrace{v_5 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}}_{\text{SI}}$

$v_6 = \begin{pmatrix} \pi \\ \sqrt{17} \\ e \end{pmatrix}$,

$$W = \{x \in \mathbb{R}^3 : x_1 + x_2 = -x_3\} \quad v_1, v_3 \in W \Rightarrow \mathrm{Span}(\{v_1, v_3\}) = W$$

$v_5 \notin W$, quindi $\{v_1, v_3, v_5\}$ sono linearmente indip.

$\Rightarrow \{v_1, v_3, v_5\}$ sono base di $\mathbb{R}^3 = V$

b) $V = \{x \in \mathbb{R}^4 : x_1 + x_2 - x_3 - x_4 = 0\} \rightarrow$, $\dim V = 3$

$$\underbrace{v_1 = \begin{pmatrix} 3 \\ -7 \\ 1 \\ -5 \end{pmatrix}}_{\text{ok}}, \quad \underbrace{v_2 = \begin{pmatrix} 2 \\ 6 \\ 9 \\ -1 \end{pmatrix}}_{\text{ok}}, \quad \underbrace{v_3 = \cancel{\begin{pmatrix} 0 \\ 32 \\ 25 \\ 7 \end{pmatrix}}}_{v_3 = -2v_1 + 3v_2}, \quad \underbrace{v_4 = \begin{pmatrix} 6 \\ -1 \\ 1 \\ 4 \end{pmatrix}}_{\text{ok}}, \quad v_5 = \begin{pmatrix} 7 \\ 8 \\ -5 \\ 20 \end{pmatrix}$$

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_4 v_4 = 0 \Leftrightarrow \lambda_1, \lambda_2, \lambda_4 = 0$ (verificato).

$\Rightarrow \{v_1, v_2, v_4\}$ lin. indip. $\Rightarrow \{v_1, v_2, v_4\}$ sono base di V.

4.5.1. Nello spazio V dato, considerare i sottospazi W e Z dati e calcolare le dimensioni di W, Z, $W+Z$, $W \cap Z$.

$$\dim(W+Z) = \dim(W) + \dim(Z) - \dim(W \cap Z)$$

$$V = \mathbb{R}^4. \quad W = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \end{pmatrix}}_{= w_1}, \underbrace{\begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}}_{= w_2} \right\}, \quad Z = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix}}_{= z_1}, \underbrace{\begin{pmatrix} -9 \\ 3 \\ 0 \\ 8 \end{pmatrix}}_{= z_2} \right\}$$

- $\dim W = \dim Z = 2$ (entrambi sono span di 2 vettori lin. indip.).

- $W \cap Z$. $\quad v \in W \cap Z, \iff \exists\, \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ t.c.

$$\left.\begin{array}{l} \bullet\ v = \lambda_1 w_1 + \lambda_2 w_2 \to v \in W \\ \bullet\ v = \mu_1 z_1 + \mu_2 \cdot z_2 \to v \in Z \end{array}\right\} \iff \underbrace{\lambda_1 w_1 + \lambda_2 w_2}_{v} \underbrace{- \mu_1 z_1 - \mu_2 z_2}_{-v} = 0$$

$\iff$

$$\begin{cases} -\lambda_1 + 2\lambda_2 - 3\mu_1 + 9\mu_2 = 0 \\ 4\lambda_1 + \lambda_2 - 2\mu_1 - 3\mu_2 = 0 \\ 2\lambda_1 - \lambda_2 - \mu_1 = 0 \\ \lambda_1 + 3\lambda_2 + \mu_1 - 8\mu_2 = 0 \end{cases} \rightarrow \text{soluzioni: } \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} = \left\{ \begin{pmatrix} 2t \\ t \\ 3t \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

Base per soluzioni e' il vettore $\begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}$. $\Rightarrow \dim(W \cap Z) = 1$

Base per $W \cap Z$: $2w_1 + w_2 - 3z_1 - z_2 = 0$

$$\underbrace{2w_1 + w_2}_{\in W} = \underbrace{3z_1 + z_2}_{\in Z} \Rightarrow 2w_1 + w_2 \in W \cap Z$$

$$2w_1 + w_2 = \begin{pmatrix} 0 \\ 9 \\ 3 \\ 5 \end{pmatrix} \rightarrow \text{generatore di } W \cap Z$$

Per $W + Z = \mathrm{Span}(\{w_1, w_2, z_1, z_2\}) \rightarrow$ estrarre una base

- $\{w_1, w_2\}$ indip. poiché base di W.
- $\{w_1, w_2, z_1\}$ sono indip. (verificato).

- $\{w_1, w_2, z_1, z_2\}$ NON sono lin. indip. $\Rightarrow \dim(W+Z) = 3$