ETA 27/11/14

$$C_n(X) = \langle \sigma : \Delta_n \to X \text{ cont.} \rangle$$

$$\varphi_i^{(n)} : \Delta_{n-1} \to \Delta_n \qquad \varphi_i^{(n)} : e_j \longmapsto \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i \end{cases}$$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \, \sigma \circ \varphi_i^{(n)}$$

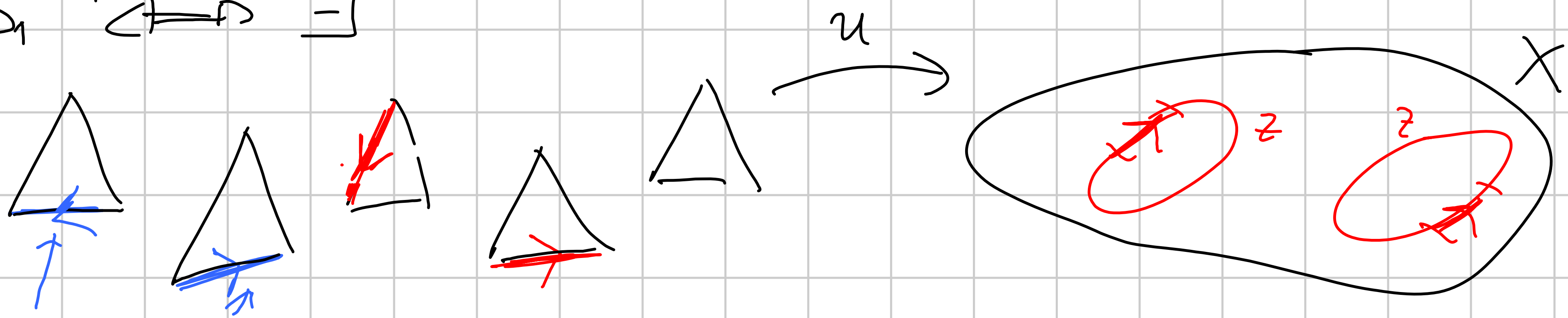
$$\partial_{n-1} \circ \partial_n = 0 \implies H_*^{SING}(X)$$

$z \in Z_n^{SING}(X) \longleftrightarrow$ mappe $z : M^{(n)} \longrightarrow X$

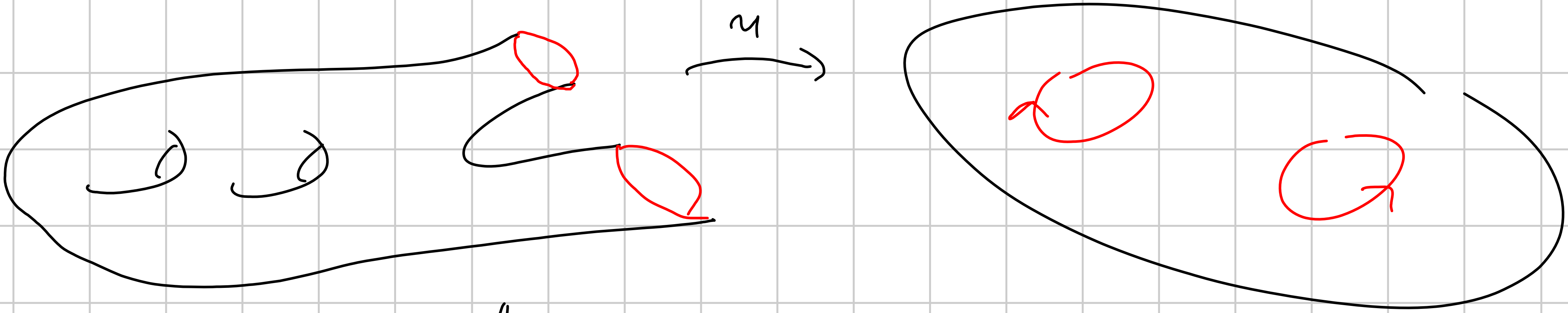
n-var con sing. in codim ≥ 3

Ora $z \in Z_1^{SING}(X) \longleftrightarrow z : \bigsqcup S^1 \longrightarrow X$

$z \in B_1^{SING} \Longleftrightarrow \exists$



$u /$ lati che non danno Z si cancellano a coppie $\Longrightarrow$ posso incollare i triangoli ottenendo



Dunque $z : \bigsqcup S^1 \to X$ ciclo è un bordo

$\Longleftrightarrow$ $z = u|_{\partial N}$ con N superficie orientata con ∂.

In generale: $z \in Z_n(X)$ visto come $z : M^{(n)} \longrightarrow X$ è in $B_n(X) \Longleftrightarrow z = u|_{\partial N}$ con $u : N^{(n+1)} \longrightarrow X$, N varietà orientabile con sing. in codim. ≥ 3.

Svantaggio: non è ovvio che $H_m^{SING}(M^{(n)}) = 0$ per $m > n$.
(però è vero perché proveremo $H^{SING} = H^{n}$).

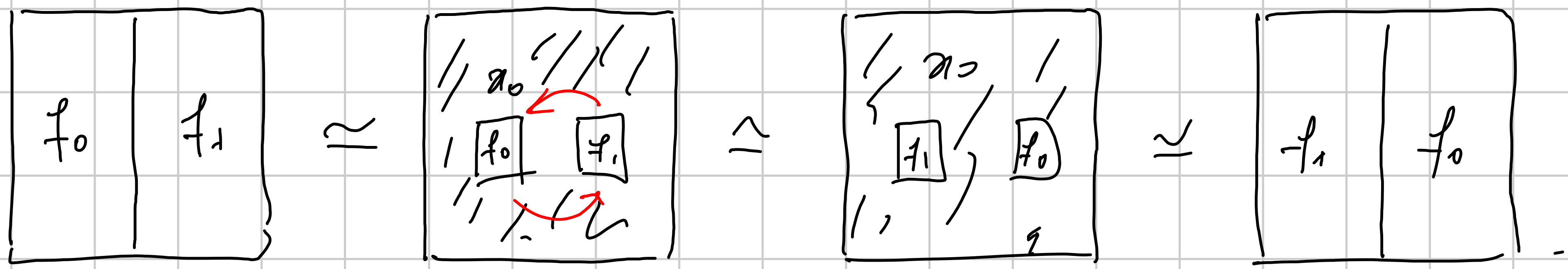
$p_0 \in S^n$ $\qquad \pi_n(X, x_0) = \dfrac{\{f : (S^n, p_0) \to (X, x_0)\}}{\simeq_{p_0}}$.

Gruppo: $I_n = [-1,1]^n$; $S^n \cong I_n / \partial I_n$ $\quad p_0 = [I_n]$

$f_0, f_1 : I_n \longrightarrow X \qquad f_0\big|_{\partial I_n} = f_1\big|_{\partial I_n} = x_0$

$f_0 \cdot f_1 =$

Oss: $n > 1 \Longrightarrow \pi_n$ è abeliano:



Fatto: $\pi_3(S^2) = \mathbb{Z}$.

Proprietà di H_*^{sing}:

(1) $H_n(X) = \bigoplus H_n(X_\alpha)$ $\quad X_\alpha$ componenti connesse p.a.

(2) $H_0(X) = \mathbb{Z}$ per X connesso p.e. $(X \neq \emptyset)$

Dim: $Z_0 = C_0 = \left\{ \sum_{i=1}^{k} m_i \cdot x_i \ : \ x_i \in X, \ m_i \in \mathbb{Z} \right\}$

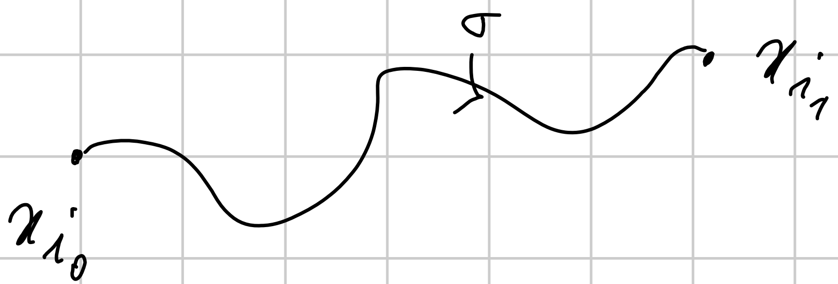
Affermo che $Z_0 \ni \sum m_i x_i \longrightarrow \sum m_i \in \mathbb{Z}$ induce $H_0 \xrightarrow{\cong} \mathbb{Z}$

- ben def: $\partial\sigma = \sigma(+1) - \sigma(-1) \longmapsto 0$
- surgettiva $m \cdot x \longmapsto m$
- iniettiva $\sum m_i = 0$; proviamo che $\sum m_i x_i$ è un bordo –

per induzione su $\sum |m_i|$; se 0, ovvio; se no trovo

$m_{i_0} < 0 < m_{i_1}$;

$\sigma : \Delta_1 \to X$



$$\Longrightarrow \left(\sum m_i \cdot x_i\right) - \partial\sigma = \sum m_i' \cdot x_i'$$

con $\sum m_i' = 0$ $\quad \sum |m_i'| = \sum |m_i| - 2$

(3) $H_n(\{pt\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{oth} \end{cases}$

$\forall n$ l'unico n-simplesso è la costante $\sigma_n : \Delta_n \to \{pt\}$;

$$\partial_n \sigma_n = \sum_{i=0}^{n} (-1)^i \underbrace{\sigma_n \circ \varphi_i^{(n)}}_{\sigma_{n-1}} = \begin{cases} 1 & n \text{ pari} \\ 0 & n \text{ dispari} \end{cases}$$

$$\begin{array}{ccccccccccccc} & & 4 & & 3 & & 2 & & 1 & & 0 & & -1 \\ (C_n^{SING}, \partial_n) = & \dots \to & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{0} & 0 \end{array}$$

$$\Rightarrow H_*^{SING} \quad \dots \quad 0 \quad 0 \quad 0 \quad \mathbb{Z}$$

(4) Si può definire $\tilde{H}_n(X)$ omologia ridotta

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & n > 0 \\ \mathbb{Z}^{k-1} & n = 0 \end{cases}$$ $k = \#$ componenti c.p.a.

ottenuta dal complesso aumentato con $C_{-1} = \mathbb{Z}$ e
$$\partial_0\left(\sum_i m_i x_i\right) = \sum m_i .$$

($\Rightarrow$ le proprietà di H_n si estendono a $\tilde{H}_n$).

(5) $f : X \to Y$ continua $\rightsquigarrow f_\# : C(X) \to C(Y)$ estende

$f_{\#_n}(\sigma) = f \circ \sigma$ con $(\sigma : \Delta_n \to X) \in C_n(X)$ _

Facile $\partial_n^Y f_\# \sigma = f_\# \partial_n^X \sigma$

$\Rightarrow f_\#$ induce $f_* : H_*^{SING}(X) \to H_*^{SING}(Y)$ _

Inoltre: $(f \circ g)_* = f_* \circ g_*$ _

(6) Omotopia: $f_0 \simeq f_1 \Rightarrow f_{0*} = f_{1*}$

(dunque $H_*^{SING}(X_0) = H_*^{SING}(X_1)$ se $X_0 \simeq X_1$).

Def: Siano $\varphi, \psi : \mathcal{C} \to \mathcal{D}$ mappe tra complessi di catene. Dico che $\varphi \simeq \psi$ se esistono $P_n : C_n \to D_{n+1}$ (prisma) t.c.

$$\begin{tikzcd}
& C_n \arrow[r, "\partial_n^{\mathcal{C}}"] \arrow[dl, "P_n"'] \arrow[d, dashed, "\varphi_n,\psi_n"] & C_{n-1} \arrow[dl, "P_{n-1}"] \\
D_{n+1} \arrow[r, "\partial_{n+1}^{\mathcal{D}}"'] & D_n &
\end{tikzcd}$$

$$\varphi_n - \psi_n = \partial_{n+1}^{\mathcal{D}} \circ P_n - P_{n-1} \circ \partial_n^{\mathcal{C}}.$$

Prop: $\varphi \simeq \psi \implies \varphi_* = \psi_*$.

Dim: sia $[z] \in H_n(\mathcal{C})$ cioè $z \in Z_n(\mathcal{C})$;

$$\varphi_{*_n}([z]) - \psi_{*_n}([z]) = [\varphi_n(z) - \psi_n(z)] =$$

$$= [\partial^{\mathcal{D}}_{n+1}(P_n(z)) - P_{n-1}(\underbrace{\partial^{\mathcal{C}}_n(z)}_{0})] = [\partial^{\mathcal{D}}_{n+1}(\dots)] = 0. \qquad \blacksquare$$

Oss: funziona anche con $\varphi_n - \psi_n = \pm \partial^{\mathcal{D}}_{n+1} \circ P_n \pm P_{n-1} \circ \partial^{\mathcal{C}}_{n-1}$.

Per provare che $f_0 \simeq f_1 \Longrightarrow f_{0_*} = f_{1_*}$ basta vedere che $f_{0_\#} \simeq f_{1_\#}$ come mappe tra complessi di catene $C(X) \to C(Y)$.

Ho $F : X \times [0,1] \to Y$ con $F\big|_{X \times \{j\}} = f_j \quad j=0,1$.

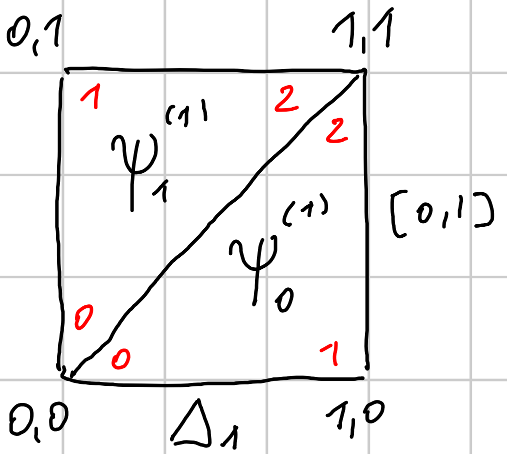
Cerchiamo $P_n : C_n(X) \to C_{n+1}(Y)$ quindi dato

$\sigma : \Delta_n \longrightarrow X$ cerchiamo $P_n(\sigma) = \begin{pmatrix} \text{comb lin. di mappe} \\ \Delta_{n+1} \to Y \end{pmatrix}$

Idea: esprimiamo $\Delta_n \times [0,1]$ come somma di $(n+1)$-simplessi

$\psi_i^{(n)} \qquad i = 0, \dots, n$

$n = 1$

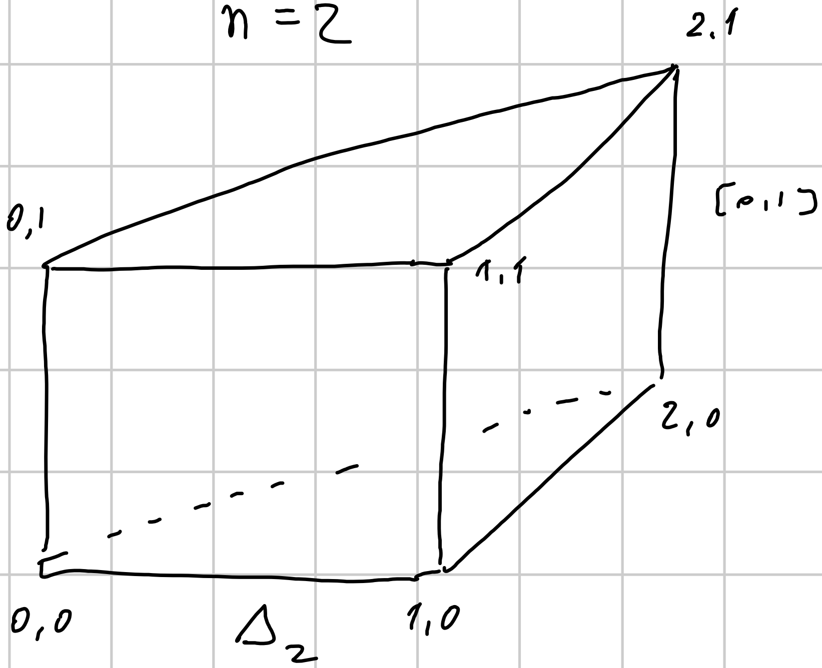


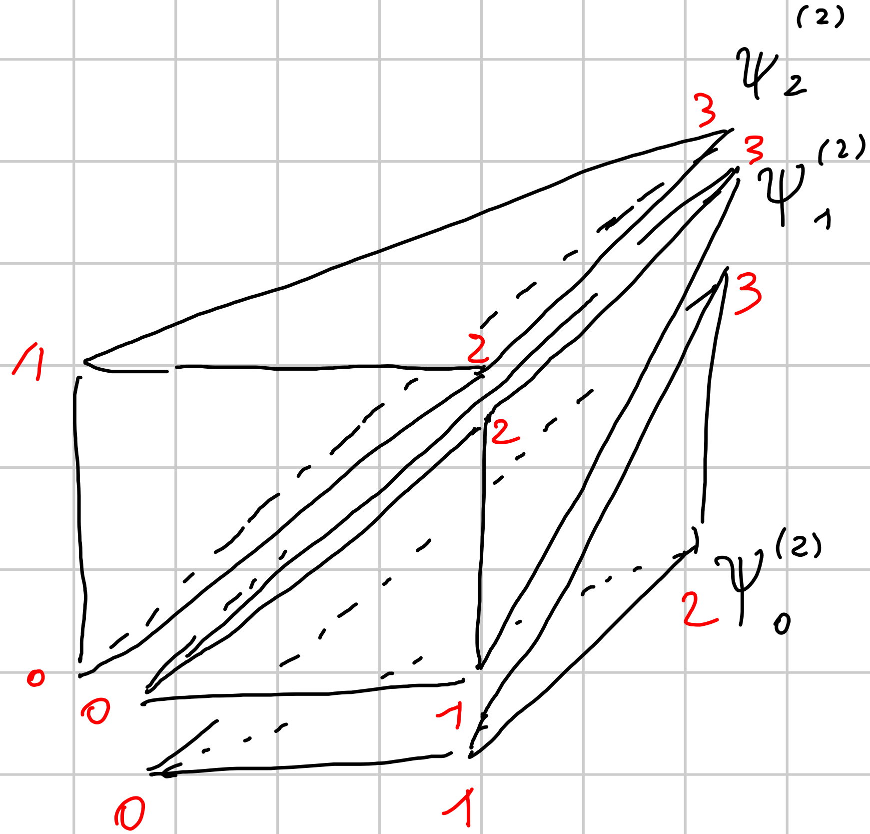
$$\psi_0^{(1)}: \quad \begin{aligned} 0 &\longmapsto 0,0 \\ 1 &\longmapsto 1,0 \\ 2 &\longmapsto 1,1 \end{aligned}$$

$$\psi_1^{(1)}: \quad \begin{aligned} 0 &\longmapsto 0,0 \\ 1 &\longmapsto 0,1 \\ 2 &\longmapsto 1,1 \end{aligned}$$

con dei segni così che venga in modo orientato, dunque le facce comuni interne si cancellano

$n = 2$





Ora si pone $P_n(\sigma) = \sum_{i=0}^{n} (-1)^{n+i} \cdot F \circ (\sigma \times id_{[0,1]}) \circ \psi_i^{(n)}$

$$\left(\overset{\curlyvee}{\partial_{m+1}} \circ \underline{P}_n\right)(\sigma) \text{ `` = '' } F \circ (\sigma \times id_{[0,1]}) \text{ calcolata su } \partial(\Delta_n \times [0,1])$$

poiché le $\psi_i^{(n)}$ e i segni sono posti in modo che sulle facce comuni interne c'è cancellazione

$$= \sigma \times id_{[0,1]} \text{ calcolata su}$$

$$\Delta_n \times \{1\} \rightsquigarrow F \circ (\sigma \times \{1\}) = f_{1\#}(\sigma)$$

$$\Delta_n \times \{0\} \rightsquigarrow - F \circ (\sigma \times \{0\}) = - f_{0\#}(\sigma)$$

$$(\partial \Delta_n) \times [0,1] \rightsquigarrow + \underline{P}_{n-1}(\partial_m \sigma)$$

$$\Longrightarrow \quad \partial_{n+1} \circ P_n = f_{1\#} - f_{0\#} + P_{n-1} \circ \partial_n$$

dunque P_n dà $f_{1\#} \simeq f_{0\#}$.

Formalmente: l'ordine dei vertici in $\psi_i^{(n)}$ sarà
$(0,0), (1,0), \dots, (n-i, 0), (n-i, 1), (n-i+1, 1), \dots, (n, 1)$. Cioè:

$$\psi_i^{(n)} : \Delta_{n+1} \to \Delta_n \times [0,1] \qquad e_j^{(n+1)} \longmapsto \begin{cases} e_j^{(n)} \times \{0\} & j = 0, \dots, n-i \\ e_{j-1}^{(n)} \times \{1\} & j = n-i+1, \dots, n+1. \end{cases}$$

e pongo $P_n(\sigma) = \sum_{i=0}^{n} (-1)^{n+1} F \circ (\sigma \times id_{[0,1]}) \circ \psi_i^{(n)}$

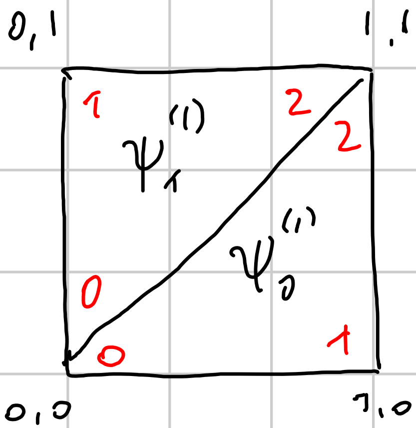
Provo:

$$\partial_{n+1} \circ \underset{\smile}{P}_n = F_{1\#} - F_{0\#} \pm \underset{\smile}{P}_{n-1} \circ \partial_n$$

Y $\qquad$ X

sui simplessi per $n = 1$.

$$\partial_2 \tau = \tau \circ \left(+ \begin{pmatrix} 0 \mapsto 1 \\ 1 \mapsto 2 \end{pmatrix} - \begin{pmatrix} 0 \mapsto 0 \\ 1 \mapsto 2 \end{pmatrix} + \begin{pmatrix} 0 \mapsto 0 \\ 1 \mapsto 1 \end{pmatrix} \right)$$



$$(\partial_2 \circ \underset{\smile}{P}_1)(\sigma) = (-1)^{1+0} \cdot \left(F \circ (\sigma \times \mathrm{id}_{[0,1]}) \right) \circ \left(+ \begin{pmatrix} 0 \mapsto 1,0 \\ 1 \mapsto 1,1 \end{pmatrix} \; \cancel{- \begin{pmatrix} 0 \mapsto 0,0 \\ 1 \mapsto 1,1 \end{pmatrix}} + \begin{pmatrix} 0 \mapsto 0,0 \\ 1 \mapsto 1,0 \end{pmatrix} \right)$$

3

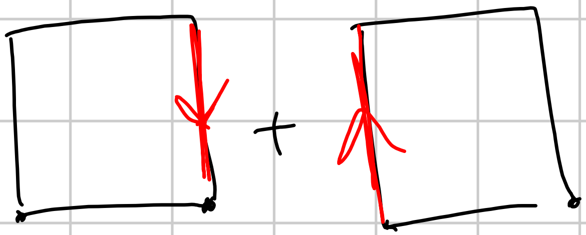
2

$$+(-1)^{1+1}\left(F\circ(\sigma\times id_{[0,1]})\right)\circ\left(\begin{array}{l} +\begin{pmatrix} 0\mapsto 0,1 \\ 1\mapsto 1,1\end{pmatrix} \quad ① \\ \cancel{-\begin{pmatrix} 0\mapsto 0,0 \\ 1\mapsto 1,1\end{pmatrix}} \\ +\begin{pmatrix} 0\mapsto 0,0 \\ 1\mapsto 0,1\end{pmatrix} \quad ④ \end{array}\right)$$

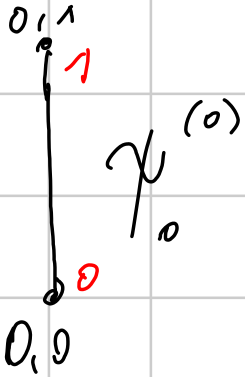
① $$F\circ(\sigma\times id_{[0,1]})\circ\begin{pmatrix} 0\mapsto 0,1 \\ 1\mapsto 1,1\end{pmatrix} = f_1\circ\sigma = f_{1\#}(\sigma)$$

② $$-F\circ(\sigma\times id_{[0,1]})\circ\begin{pmatrix} 0\mapsto 0,0 \\ 1\mapsto 1,0\end{pmatrix} = -f_0\circ\sigma = -f_{0\#}(\sigma)$$

③ + ④ $$F\circ(\sigma\times id_{[0,1]})\circ\Bigg($$



$$\Bigg) = \pm P_1(\partial\sigma)$$



In generale è complicato ma non difficile _

(7) <u>Versione relativa di</u> H^{SING} : $A \subset X$ qualsiasi sottoinsieme

$$C_n(X,A) = C_n(X) \big/ C_n(A)$$

se $p_n : C_n(X) \to C_n(X,A)$ è la proiez.

Dunque $\partial_n^{(X,A)}(p_n(\sigma)) = p_{n-1}(\partial_{n-1}^X(\sigma))$; ben def:

$$p_n(c) = 0 \Rightarrow c \in C_n(A) \Rightarrow \partial_n^X(c) \in C_{n-1}(A)$$
$$\Rightarrow p_{n-1}(\partial_n^X(c)) = 0 \,.$$

Ovviamente $\partial_{n-1}^{(X,A)} \circ \partial_n^{(X,A)} = 0 \Rightarrow H_*^{SING}(X,A)$.

Inoltre: $0 \longrightarrow C(A) \xrightarrow{i} C(X) \xrightarrow{p} C(X,A) \longrightarrow 0 \Rightarrow$

(8) LES:

$$\dots \to H_n(A) \xrightarrow{i_{*n}} H_n(X) \xrightarrow{p_{*n}} H_n(X,A) \xrightarrow{d_n} H_{n-1}(A) \to \dots$$

Fatto: $H_n(X,A) = Z_n'(X,A) / B_n'(X,A)$

$$Z_n'(X,A) = \{ c \in C_n(X) : \partial_n c \in C_{n-1}(A) \}$$

$$B_n'(X,A) = \{ \partial d + e : d \in C_{n+1}(X), \ e \in C_n(A) \}$$

e con questa interpretazione $d_n([c]) = [\partial_n(c)]$

$c \in Z'(X,A)$ $\quad \partial_n c \in Z_{n-1}(A)$ _

(9) Omotopia relativa:

$f_0, f_1 : (X,A) \to (Y,B)$

$f_0 \underset{A}{\simeq} f_1$ se esiste $F : X \times [0,1] \to Y$ t.c.

$F|_{X \times \{i\}} = f_i \quad F(A \times [0,1]) \subset B$

$\Rightarrow f_{0*} = f_{1*}$

(8½) $f:(X,A)\to(Y,B)$ cioè $f:X\to Y$, $f(A)\subset B$

$\Rightarrow f_\#: C(X)\to C(Y)$

soddisfa $f_\#(C(A))\subset C(B)$

$\Rightarrow f_\#$ induce $f_\#: C_n(X,A)\to C_n(Y,B)$

$\Rightarrow$ induce $f_*: H_n(X,A)\to H_n(Y,B)$.

(di nuovo (8)): provo che $f_{0*}=f_{1*}$ come sopra dimostrando che $f_{0\#}\simeq f_{1\#}$ infatti

$P_n^X : C_n(X) \to C_{n+1}(Y)$

soddisfa $P_n^X(C_n(A)) \subset C_{n+1}(B)$

$\Rightarrow P_n^X$ induce $P_n^{(X,A)} : C_n(X,A) \to C_{n+1}(Y,B)$

e $P_n^{(X,A)}$ definisce l'omotopia $f_{0\#} \simeq f_{1\#}$ -

(10) Successione esatta di una terna $A \subset B \subset X$:

ho esatte corte:

$$0 \to C_n(B,A) \xrightarrow{i} C_n(X,A) \xrightarrow{p} C_n(X,B) \to 0$$

$$\Rightarrow \ldots \to H_n(B,A) \xrightarrow{i_{*n}} H_n(X,A) \xrightarrow{p_{*n}} H_n(X,B) \to H_{n-1}(B,A) \to \ldots$$

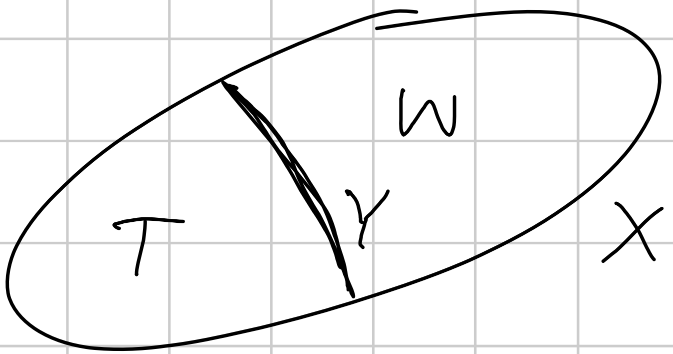
$$\overbrace{H_n(X,B) \to H_{n-1}(B) \longrightarrow H_{n-1}(B,A)}^{=}$$

in LES für (X,B)

in LES für (B,A)

(11) <u>Escissione</u>

<u>Caso PL</u> : $X = T \cup W \qquad Y = T \cap W$ <u>sottocomplessi</u>



$$\Rightarrow H_*^{PL}(X,T) \cong H_*^{PL}(W,Y)$$

sto tagliando $T \setminus Y$ sia da X
sia da T

<u>Caso SING</u> : Se $Z \subset A \subset X$ e $\overline{Z} \subset \mathrm{int}(A)$

allora

$$H_n(X,A) \cong H_n(X \setminus Z, A \setminus Z).$$