ETA 18/12/14

Visto: H_* determina $H^*(\cdot\,; G)$.

Ma: Su H^* c'è struttura di prodotto.

Def: Fisso R anello commut. con 1.
Opero nella teoria singolare. Pongo:

$$\cup : C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R)$$

$$(\varphi \cup \psi)([v_0, \dots, v_{k+\ell}]) = \\ = \varphi([v_0, \dots, v_k]) \cdot \psi([v_k, \dots, v_{k+\ell}]).$$

$[\cdot]$ = abbreviazione $\sigma : [v_0, \dots, v_{k+\ell}] \to X$
e delle sue restrizioni

Lem: $\delta_{k+\ell}(\varphi \cup \psi) = (\delta_k \varphi) \cup \psi + (-1)^k \varphi \cup (\delta_\ell \psi).$

Dimo: laboriosa ma non difficile.

Esempio: $k = l = 1$.

$\varphi \in C^1$, $\psi \in C^1$, $\varphi \cup \psi \in C^2$, $\delta_2(\varphi \cup \psi) \in C^3$.

$$\delta_2(\varphi \cup \psi)(0123) = (\varphi \cup \psi)(\partial_3(0123)) =$$

$$= (\varphi \cup \psi)(123 - 023 + 013 - 012)$$

$$= \underbrace{\varphi(12)\cdot\psi(23)}_{1} - \underbrace{\varphi(02)\cdot\psi(23)}_{2} + \underbrace{\varphi(01)\cdot\psi(13)}_{3} - \underbrace{\varphi(01)\cdot\psi(12)}_{4}$$

$$\big((\delta_1\varphi)\cup\psi - \varphi\cup(\delta_1\psi)\big)(0123) = (\delta_1\varphi)(012)\cdot\psi(23) - \varphi(01)(\delta_1\psi)(123)$$

$$= \varphi(\partial_2(012))\cdot\psi(23) - \varphi(01)\cdot\psi(\partial_2(123))$$

$$= \varphi(12-02+01)\cdot\psi(23) - \varphi(01)\cdot\psi(23-13+12)$$

$$= \underbrace{\varphi(12)\cdot\psi(23)}_{1} - \underbrace{\varphi(02)\cdot\psi(23)}_{2} + \cancel{\varphi(01)\cdot\psi(23)}$$

$$- \cancel{\varphi(01)\cdot\psi(23)} + \underbrace{\varphi(01)\cdot\psi(13)}_{3} - \underbrace{\varphi(01)\cdot\psi(12)}_{4} .$$

<u>Cor</u>: (1) $Z^k \cup Z^\ell \subset Z^{k+\ell}$

(2) $B^k \cup Z^\ell \subset B^{k+\ell}$

(3) $Z^k \cup B^\ell \subset B^{k+\ell}$

<u>Dim</u>: $\delta_k \varphi = 0 \quad \delta_\ell \varphi = 0$

$$\delta_{k+\ell}(\varphi \cup \psi) = (\underbrace{\delta_k \varphi}_{0}) \cup \psi + (-1)^k \varphi \cup (\underbrace{\delta_\ell \psi}_{0}) = 0$$

(2) $\varphi \in B^k$ i.e. $\varphi = \delta_{k-1}\eta$, $\psi \in Z^\ell$, $\delta_\ell \psi = 0$

$$\delta_{k+\ell-1}(\eta \cup \psi) = (\underbrace{\delta_{k-1}\eta}_{\varphi}) \cup \psi + (-1)^{k-1} \eta \cup (\underbrace{\delta_\ell \psi}_{0}) = \varphi \cup \psi$$

(3) analogo. ▨

<u>Cor</u>: $\cup$ induce una mappa $H^k \times H^\ell \to H^{k+\ell}$.

Proprietà:

- associativo: $(\varphi \cup \psi) \cup \eta \; ([v_0, \dots, v_{k+l+m}])$
$$= \varphi \cup (\psi \cup \eta) \; ([v_0, \dots, v_{k+l+m}])$$
$$= \varphi([v_0, \dots, v_k]) \cdot \psi([v_k, \dots, v_{k+l}]) \cdot \eta([v_{k+l}, \dots, v_{k+l+m}])$$

- distributivo
- ha $1 \in H^0(X; R)$ (X connesso)

$$(1 \cup \psi ([v_0, \dots, v_\ell]) = \underbrace{1(v_0)}_{=1} \cdot \psi([v_0, \dots, v_\ell])$$

- $$H^* = \bigoplus_{k=0}^{+\infty} H^k(X;R)$$ anello graduato

e $\cup$ è commutativo nel senso degli anelli graduati:

$$\varphi \in H^k, \psi \in H^\ell \Rightarrow$$
$$\psi \cup \varphi = (-1)^{k \cdot \ell} \varphi \cup \psi$$

<u>Dimo</u>: Definisco $\rho_n : C_n(X) \circlearrowleft$ estendendo

$$\rho_n([v_0, \dots, v_n]) = \varepsilon_n \cdot [v_n, \dots, v_0] \qquad \varepsilon_n = (-1)^{n(n+1)/2}.$$

Noto che la mappa
$$\Delta_n \to \Delta_n$$
$$e_j \longmapsto e_{n-j}$$
è omotopa a $\varepsilon_n \cdot id$. Infatti la matrice di cambio di base da

$e_1 - e_0, \dots, e_n - e_0$ a $e_{n-1} - e_n, \dots, e_0 - e_n$ è

$$\begin{pmatrix} 0 & & & & \cdots & 1 & 0 \\ \vdots & & 1 & \cdots & & & \vdots \\ \vdots & 1 & 0 & & & & \vdots \\ 1 & 0 & 0 & & & & 0 \\ -1 & -1 & -1 & \cdots & & & -1 \end{pmatrix}$$ che ha $\det = \varepsilon_n$.

Ne segue che ρ_n è omotopa a $id_{C_*(X)}$

$\Rightarrow$ induce l'identità in H_* e H^*.

$$(\psi \cup \varphi)([v_0, \dots, v_{k+l}]) = \varepsilon_{k+l} (\psi \cup \varphi)([v_{k+l}, \dots, v_0])$$

$$= \varepsilon_{k+\ell} \cdot \psi([v_{k+\ell},\dots,v_k]) \cdot \varphi([v_k,\dots,v_0])$$

$$= \varepsilon_{k+\ell} \cdot \varepsilon_\ell \cdot \psi([v_k,\dots,v_{k+\ell}]) \cdot \varepsilon_k \cdot \varphi([v_0,\dots,v_k])$$

$$= \underbrace{\varepsilon_{k+\ell} \cdot \varepsilon_k \cdot \varepsilon_\ell}_{\|} \cdot \underbrace{\varphi([v_0,\dots,v_k]) \cdot \psi([v_k,\dots,v_{k+\ell}])}_{(\varphi \cup \psi)([v_0,\dots,v_{k+\ell}])}$$

$$(-1)^{\frac{1}{2}\left((k+\ell)(k+\ell+1) + k(k+1) + \ell(\ell+1)\right)}$$

$$= (-1)^{\frac{1}{2}\left(k(k+1) + k\cdot\ell + \ell(\ell+1) + \ell k + k(k+1) + \ell(\ell+1)\right)}$$

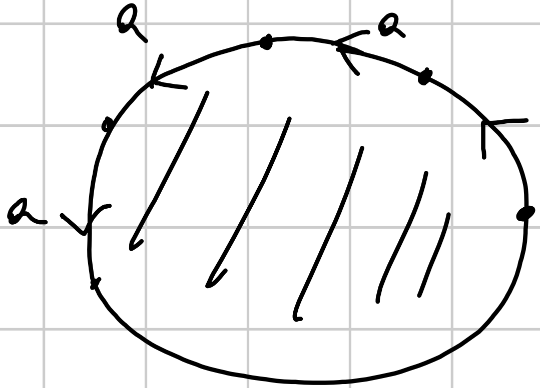
$$= (-1)^{k\cdot\ell}. \qquad \blacksquare$$

Esempio (uso la teoria simpliciale):

$$X = \{ z \in \mathbb{C} : |z| \leq 1 \} \Big/ {\scriptstyle z \sim w}$$

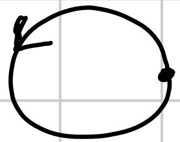
se $|z| = |w| = 1$

e $z^m = w^m$

$\Longrightarrow$ $X =$ 

ovvero X è ottenuto attaccando D^2 a S^1

con $S^1 \ni z \longmapsto z^m \in S^1$



dunque ho celle c_0, c_1, c_2 e $\partial c_1 = 0$

e $\partial c_2 = m \cdot c_1$

$H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}/m$, $H_2 = 0$

$\Rightarrow H^0 = \mathbb{Z}$, $H^1 = 0$, $H^2 = \mathbb{Z}/m$

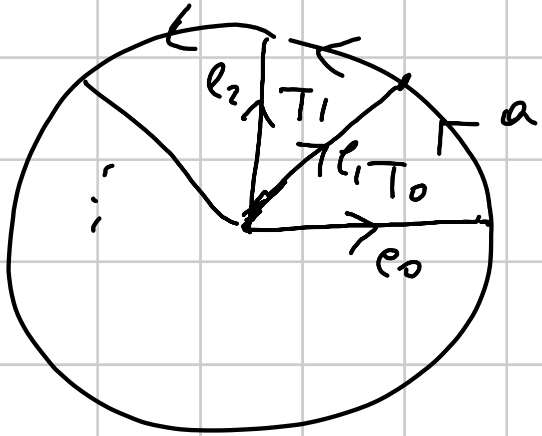
Invece a coeff. in $\mathbb{Z}/m$:

$$H^0_{\mathbb{Z}/m} = \mathbb{Z}/m$$

$$H^1_{\mathbb{Z}/m} = \underbrace{\mathrm{Hom}(\mathbb{Z}/m, \mathbb{Z}/m)}_{\mathbb{Z}/m} \oplus \underbrace{\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/m)}_{0}$$

$$H^2_{\mathbb{Z}/m} = \underbrace{\mathrm{Hom}(0, \mathbb{Z}/m)}_{0} \oplus \underbrace{\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/m)}_{\mathbb{Z}/m}.$$

Triangoliamo nel senso dei Δ-complessi:



$$H^0_{\mathbb{Z}/m} = \mathbb{Z}/m = \langle 1 \rangle \qquad H^1_{\mathbb{Z}/m} = \mathbb{Z}/m = \langle \hat{a} \rangle \qquad H^2_{\mathbb{Z}/m} = \mathbb{Z}/m = \langle \hat{D} \rangle$$

l'unico prodotto da calcolare è $\hat{a} \vee \hat{a}$.

Cerco una cocatena che rappresenti $\hat{a}$: voglio

$\hat{a}(a) = 1$ e $\delta_1 \hat{a} = 0$ dunque

$(\delta_1 \hat{a})(T_j) = 0 \quad \forall j \qquad \hat{a}(\partial_2 T_j) = 0 \quad \forall j$

$\hat{a}(e_j + a - e_{j+1}) = 0 \implies \hat{a}(e_{j+1}) = 1 + \hat{a}(e_j)$

Prendo allora $\hat{a}(e_j) = j$. Allora

$\hat{a} \cup \hat{a} = \hat{a} \cup \hat{a}(D) = \hat{a} \cup \hat{a}(T_0 + \dots + T_{m-1})$

$$= 0 + 1 + \dots + m-1 = \frac{m(m-1)}{2} \in \mathbb{Z}/m$$

Se m è dispari viene 0; invece se m è pari è

$m = 2k$ $$\frac{2k(2k-1)}{2} = k(2k-1) = -k = k \in \mathbb{Z}/_{2k}$$

Non è una sorpresa: sappiamo

$$\underset{\underset{H^1}{\cap}}{\hat{a}} \cup \underset{\underset{H^1}{\cap}}{\hat{a}} = (-1)^{1 \cdot 1} \hat{a} \cup \hat{a} = - \hat{a} \cup \hat{a}.$$

$$\Rightarrow 2 \cdot \hat{a} \cup \hat{a} = 0$$

n dispari $\Longrightarrow$ $\hat{a} \cup \hat{a} = 0$

$n = 2k$ pari $\quad \hat{a} \cup \hat{a} \in \{0, k\}$

viene non banale.

Caso part. con $n = 2$ $\Longrightarrow$ $X = \mathbb{P}^2(\mathbb{R})$

0	1	2
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

e il quadrato del generatore di $\mathbb{Z}/2$ in grado 1 è il generatore di $\mathbb{Z}/2$ in grado 2

$$\Longrightarrow H^*(\mathbb{P}^2(\mathbb{R});\mathbb{Z}/2) \underset{\text{come algebra}}{\cong} \mathbb{Z}/2[u]\big/u^3 .$$

Dualità di Poincaré :

<u>Teo</u> : $M^{(n)}$ varietà PL chiusa ;

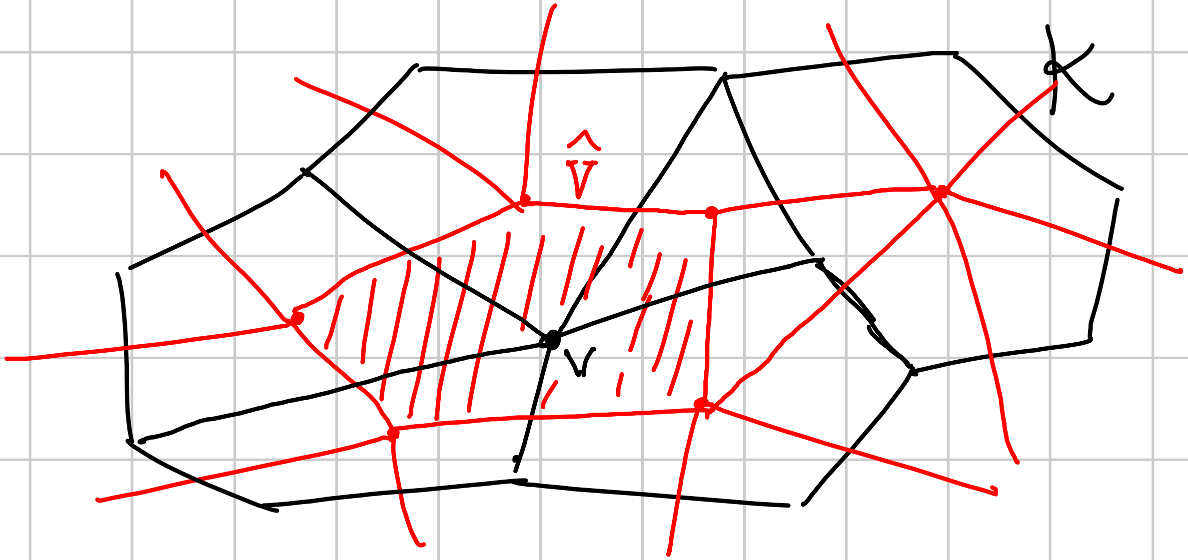
M orientata $\Rightarrow H_p(M) \cong H^{n-p}(M)$

Sempre: $H_p(M;\mathbb{Z}/2) \cong H^{n-p}(M;\mathbb{Z}/2)$.

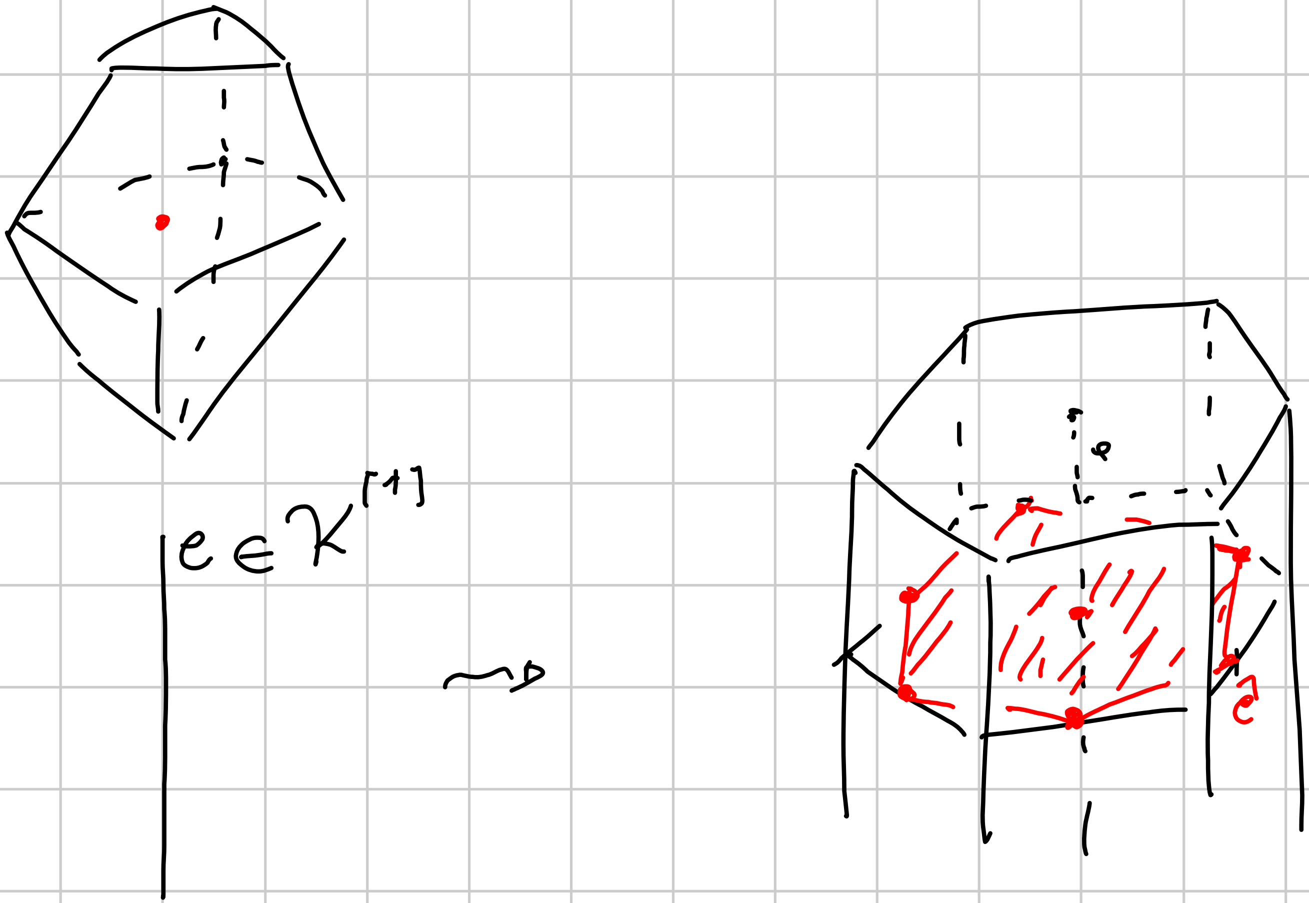
(Combinandolo con UCT si trova ad es che
$\mathrm{rank}(H_{n-p}(M)) = \mathrm{rank}(H_p)$
da cui segue che $\chi(M^{(n)}) = 0$ se n è dispari.)

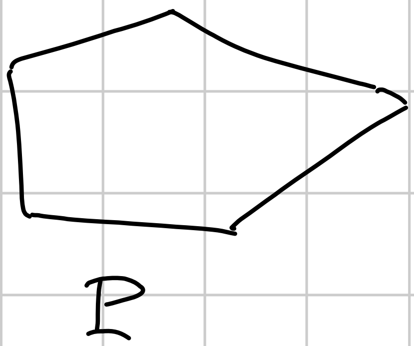
Dimo: suppongo $M = |K|$ K complesso politopale. Definisco complesso politopale duale $\hat{K}$ così: prendo un vertice $\hat{\sigma}$ per ogni $\sigma \in K^{[n]}$. Per ogni $\tau \in K$ ho una $\hat{\tau} \in \hat{K}$ i cui vertici sono i $\hat{\sigma}$ al variare di $\sigma \in K^{[n]}$ con $\sigma \supset \tau$.

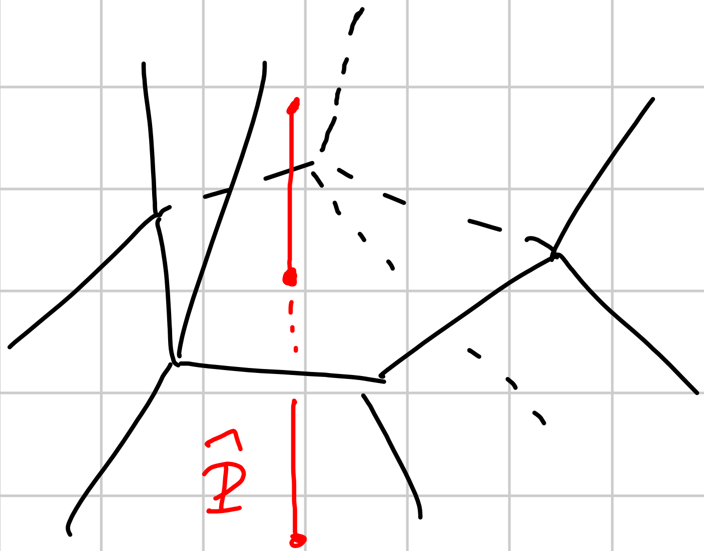
$n=2$



$n = 3$







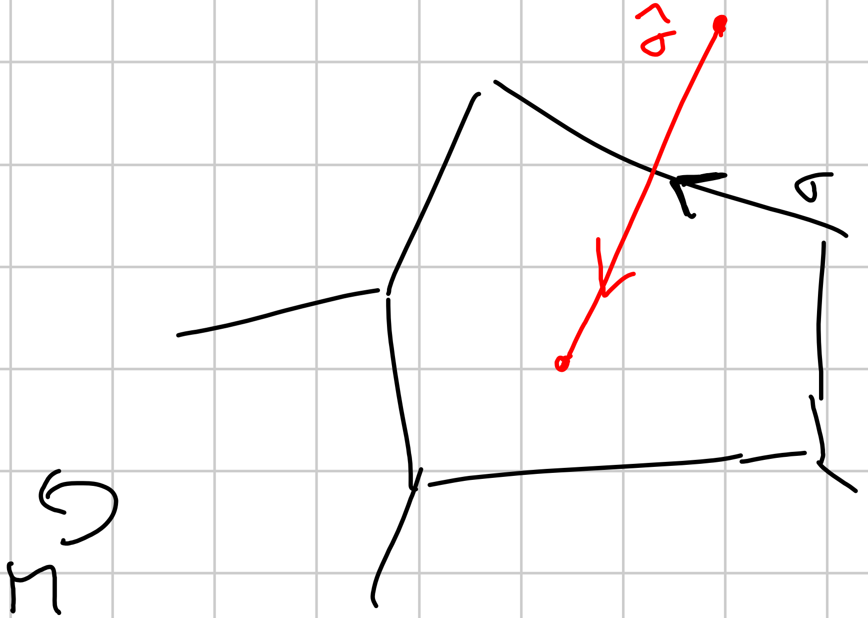
Per costruz. $P \pitchfork \hat{P} = \{pt\}$.

Nel caso di M orientabile oriento i

vertici di K positivi e $K^{[n]}$ come M;

ora oriento ogni $\hat{\sigma}$ in modo che

(base pos. di σ, base pos. di $\hat{\sigma}$) = (base pos. di M)



Per $\dim\sigma = n$ vuol dire che prendo i vertici di $\hat{K}$

Per $\dim\sigma = 0$ vuol dire che prendo su $\hat{K}^{[n]}$ l'orientaz. di M

Indico con $\bar{\sigma}$ il duale algebrico di σ – Definisco:

$$\varphi_p : C_p(\hat{K}) \xrightarrow{\cong} C^{n-p}(K)$$

$$\hat{\sigma} \longmapsto \bar{\sigma} \qquad \forall \sigma \in K^{[n-p]}$$

Affermo che φ_p trasforma ∂ in δ, il che dà la conclusione. Formalmente:

$$\varphi_{p-1} \circ \partial_p = \delta_{m-p} \circ \varphi_p .$$

Calcoliamo per $\sigma \in K^{[n-p]}$, $\tau \in K^{[n-p+1]}$

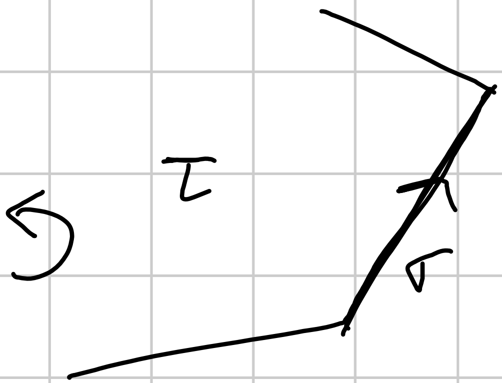
$$\delta_{m-p}(\varphi_p(\hat{\sigma})) : \quad \tau \longmapsto \bar{\sigma}(\partial_{m-p+1}\tau) = \begin{cases} \pm 1 & \text{se } \sigma \subset \partial\tau \\ 0 & \text{altrimenti} \end{cases}$$

$$\varphi_{p-1}(\partial_p \hat{\sigma}) : \quad \tau \longmapsto \sum_{\substack{\eta \subset \partial_p \hat{\sigma} \\ \eta \in \hat{K}^{[p-1]}}} \pm \bar{\eta}(\tau) = \begin{cases} \pm 1 & \text{se } \hat{\tau} \subset \partial\hat{\sigma} \\ 0 & \text{altrimenti} \end{cases}$$

Ora $a \subset b \iff \hat{b} \subset \hat{a}$ quindi su $\mathbb{Z}/2$ è OK. Su $\mathbb{Z}$ restano da verificare i segni:

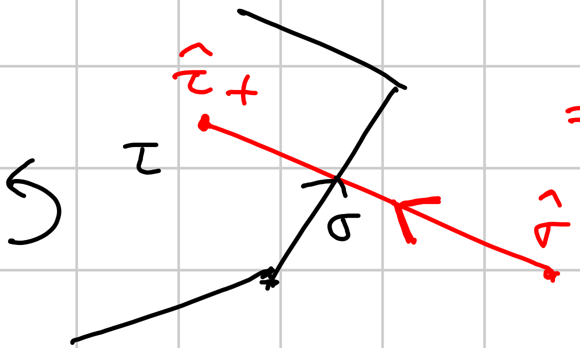
esempi:

$n = 2,\ p = 1$



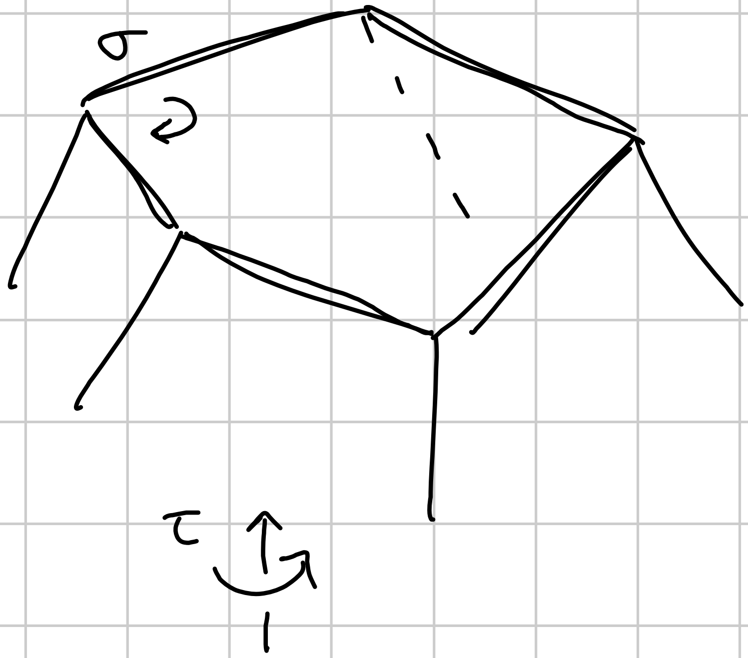
$\Longrightarrow$

cioè $\partial \tau \supset + \sigma$

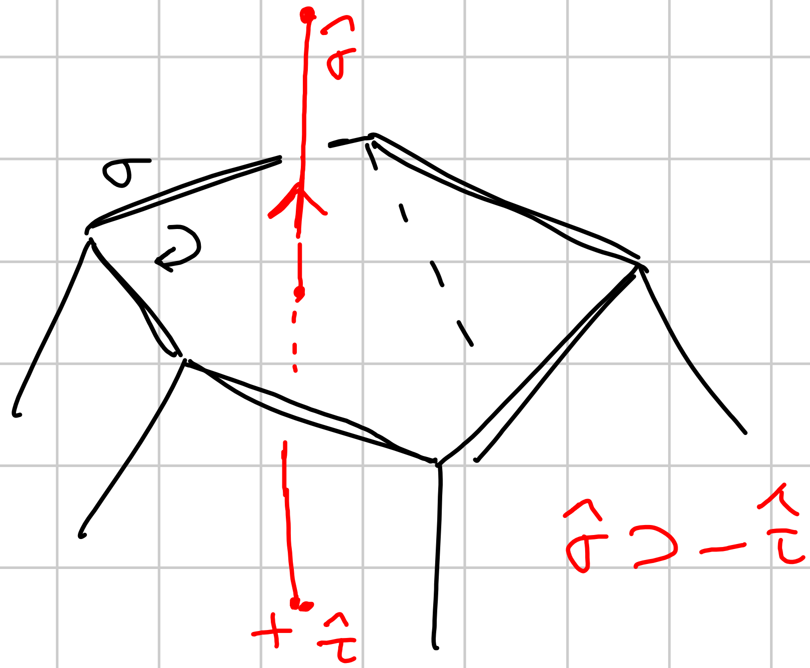


$\Longrightarrow \hat{\sigma} \supset + \hat{\tau}$

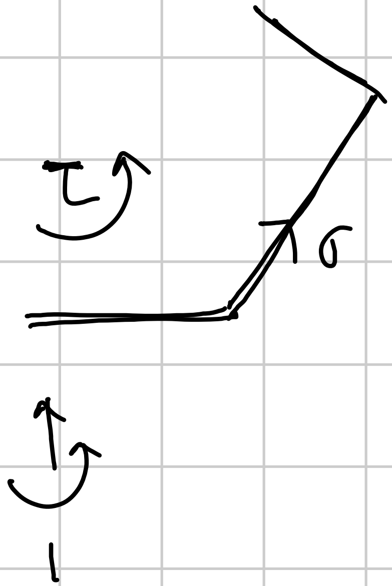
$n=3, p=1$



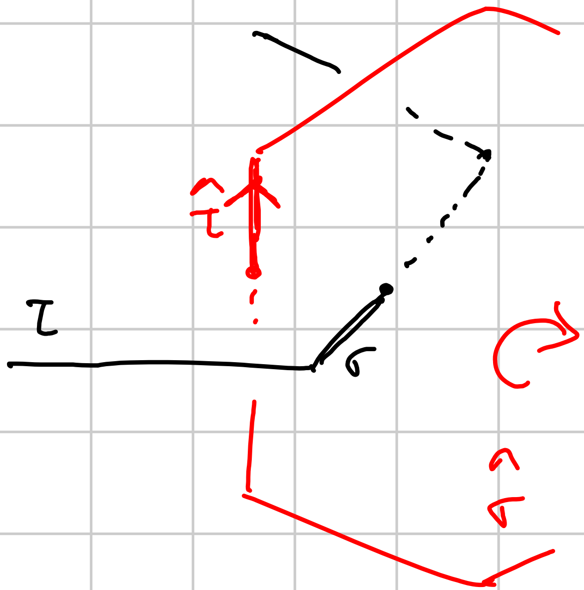
$\partial\tau \supset -\sigma$



$n=3, p=2$



$\Longrightarrow$



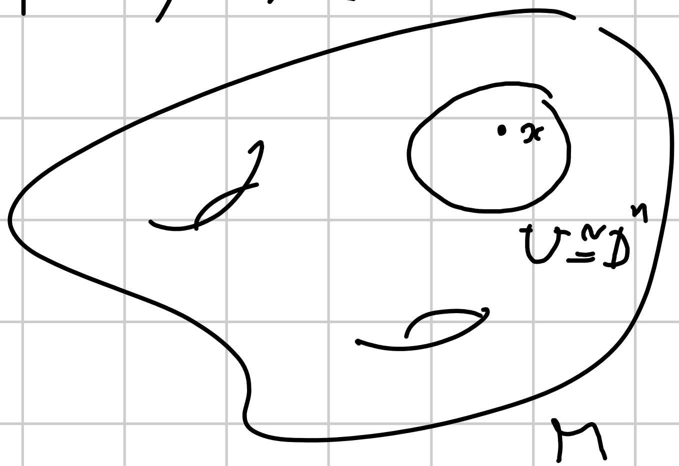
$\partial\langle\tau\rangle+\sigma$

$\Longrightarrow \partial\langle\hat{\sigma}\rangle+\hat{\tau}\,.$ $\blacksquare$

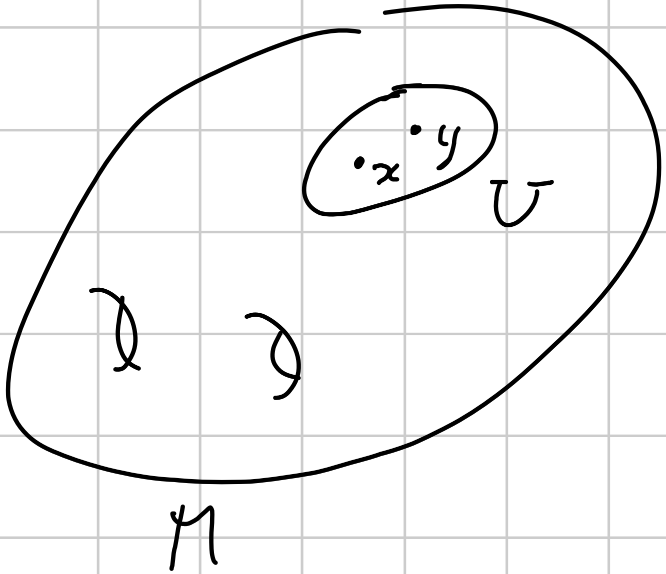
<u>Oss</u>: sia M una n-var. TOP, $x \in M$:

$$H_n(M, M \setminus \{x\}) \underset{\text{escissione}}{\cong} H_n(U, U \setminus \{x\})$$

$$\cong H_n(D^n, D^n \setminus \{0\}) \underset{\text{omotopia}}{\cong} H_n(D^n, S^{n-1}) = \mathbb{Z}$$



Def: Una orientaz. per $M^{(n)}$ TOP (connessa) è la scelta di un generatore per ogni $H_n(M, M \setminus \{x\})$ loc. coerente, cioè:



$$\begin{array}{cc}
H_n(M, M \setminus \{x\}) & +1 \\
\downarrow \cong & \Big\downarrow \\
H_n(U, \partial U) & \\
\uparrow \cong & \\
H_n(M, M \setminus \{y\}) & +1
\end{array}$$

<u>Def</u> (in teoria SING) : $\quad k \geq p$

$$\cap : C_k(X) \times C^p(X) \to C_{k-p}(X)$$

$$\sigma \cap \varphi = \varphi\left(\sigma|_{[e_0,\dots,e_p]}\right) \cdot \underbrace{\sigma|_{[e_p,\dots,e_k]}}$$

tacitamente precomposto con
$[e_0,\dots,e_{k-p}] \to [e_p,\dots,e_k]$

Lem: $\partial_{k-p}(\sigma \frown \varphi) = (-1)^p \left((\partial_k \sigma) \frown \varphi - \sigma \frown (\delta_p \varphi) \right)$

Dimo laboriosa ma non difficile _ Esempio $k=2$, $p=1$

$$\partial_1(\sigma \frown \varphi) = \partial_1\left(\varphi\left(\sigma|_{[e_0,e_1]}\right) \cdot \sigma|_{[e_1,e_2]} \right)$$
$$= \varphi\left(\sigma|_{[e_0,e_1]}\right) \cdot \left(\sigma(e_2) - \sigma(e_1)\right)$$
$$= \varphi(\sigma_{01}) \cdot \sigma_2 - \varphi(\sigma_{01}) \cdot \sigma_1$$

$$-(\partial_2 \sigma) \frown \varphi + \sigma \frown (\delta_1 \varphi) =$$

$$= -(\sigma_{12} - \sigma_{02} + \sigma_{01}) \cap \varphi + \delta_1 \varphi(\sigma) \cdot \sigma(e_2)$$

$$= -\varphi(\sigma_{12}) \cdot \sigma_2 + \varphi(\sigma_{02}) \cdot \sigma_2 - \varphi(\sigma_{01}) \cdot \sigma_1$$

(finire per esercizio).

Come nel caso $\cup$ il lemma comporta che è ben def

$$\cap : H_k(X) \times H^p(X) \longrightarrow H_{k-p}(X).$$

Fatto: se $M^{(n)}$ TOP è orientata ho un generatore canonico $[M]$ di $H_n(M;\mathbb{Z})$; quello per cui

$$0=H_n(M\setminus\{x\}) \to \overset{\mathbb{Z}}{\overset{\shortparallel}{H_n(M)}} \longrightarrow H_n(M,M\setminus\{x\})=\mathbb{Z}$$

$$+1 \longmapsto +1$$

Se M non è orientata ho $H_2(M;\mathbb{Z}/2)\cong\mathbb{Z}/2$.

Dualità di Poincaré (TOP):

Teo: $H^p(M) \longrightarrow H_{n-p}(M)$

$\varphi \longmapsto [M] \cap \varphi$

è un isomorfismo (su $\mathbb{Z}$ se M è orient, su $\mathbb{Z}/2$ sempre) —

Idea del fatto che è lo stesso di prima: caso politopale era

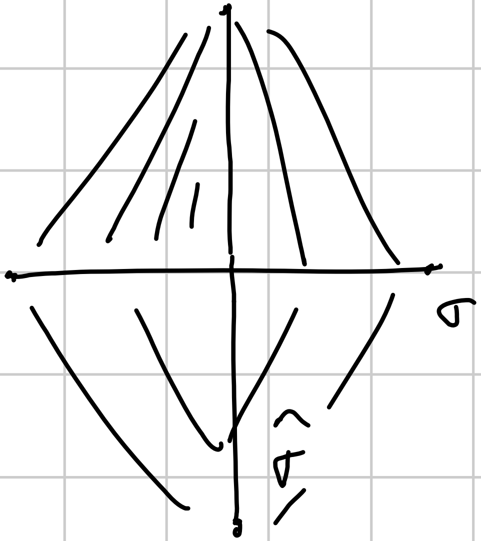
$$H^p(M) \longrightarrow H_{n-p}(M)$$
$$\bar{\sigma} \longmapsto \hat{\sigma}$$

duale alg. di $\sigma \in K^{[p]}$ duale geom $\in \hat{K}^{[n-p]}$

Ora si ha che $[M] \cap \bar{\sigma} = \hat{\sigma}$:

idea

$[M]$ contiene il join $\sigma \cdot \hat{\sigma}$



$$\Longrightarrow [M] \cap \bar{\sigma} =$$
$$= (\sigma \cdot \hat{\sigma}) \cap \bar{\sigma}$$
$$= \bar{\sigma}(\sigma) \cdot \hat{\sigma} = \hat{\sigma}. \qquad \blacksquare$$