Geometrie 29/5/14

(11) $d\omega_0 = d\omega_1$ su $\mathbb{R}^2 \Rightarrow$?

$d(\omega_0 - \omega_1) = 0$ su $\mathbb{R}^2$ + $\mathbb{R}^2$ s.c.

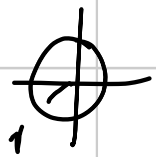
$\Rightarrow \omega_0 - \omega_1 = dU \qquad U : \mathbb{R}^2 \to \mathbb{R}$

(12) $\displaystyle\int_\alpha \cos(y)\,(dx + dz) - (x+z)\sin(y)\,dy$

$\displaystyle = \int_\alpha d\Big((x+z)\cdot \cos y \Big) = (x+z)\cdot \cos(y) \Big|_{\alpha(0)}^{\alpha(1)}$

$$= (x+z)\cdot\cos(y)\Big|_{(0,0,0)}^{(\pi,0,0)} = \pi$$

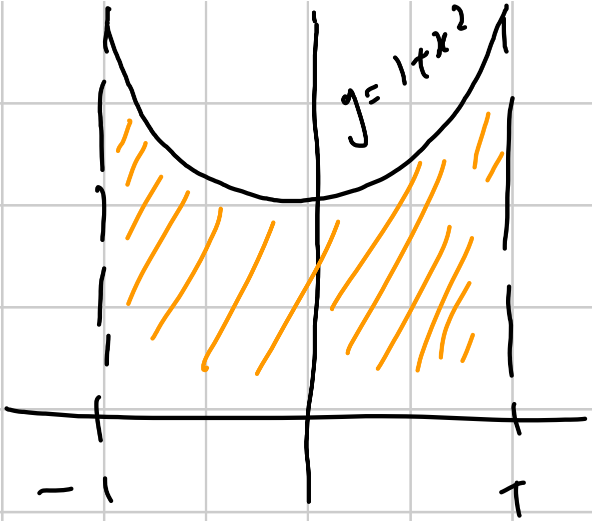
(13) $$\int_{\partial\Delta} \left(\cos(e^y)\,dy - y\,dx\right) = \int_{\Delta} d\left(\cos(e^y)\,dy - y\,dx\right)$$

$\Delta =$ 

$$= \int_{\Delta} -dy\,dx = \int_{\Delta} dx\,dy$$

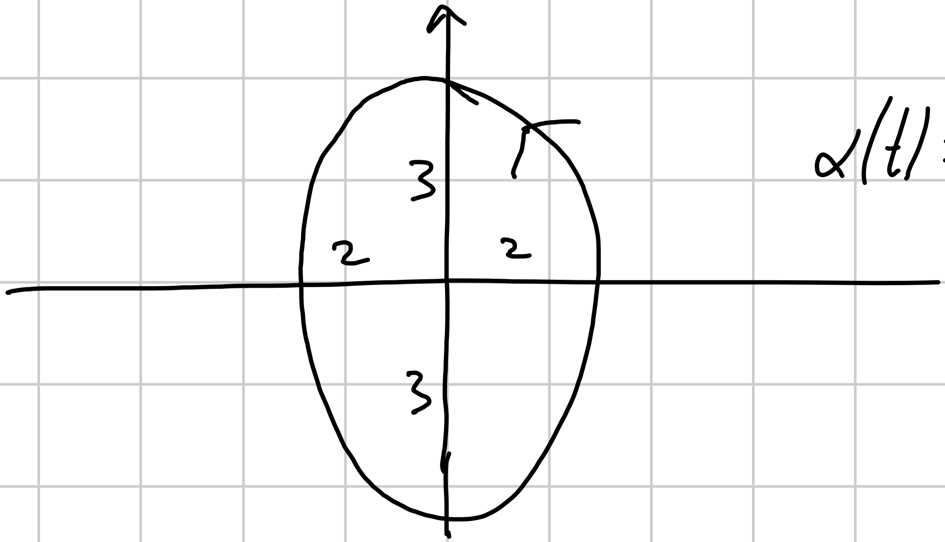
$$= \text{Area}(\Delta) = \pi$$

(14) $$\int_{\partial A} x\,dy = \int_{A} dx\,dy = \text{Area}(A) = \int_{-1}^{1} (1+x^2)\,dx$$



$$= 2 + \frac{2}{3} = \frac{8}{3}.$$

(15)



$\alpha(t) = (2\cos(t), 3\sin(t))$

$t \in [0, 2\pi]$

$\boxed{16}$ (7) Trovare $k \in \mathbb{R}$ t.c.

$$\frac{2xy^2(y\,dx + kx\,dy)}{1 + x^2y^3} \quad \text{sia chiusa.}$$

- Imporre $\displaystyle \frac{\partial}{\partial x}\left(\frac{2kx^2y^2}{1+x^2y^3}\right) = \frac{\partial}{\partial y}\left(\frac{2xy^3}{1+x^2y^3}\right) \ \dots$

- $\displaystyle \underbrace{\frac{2xy^3}{1+x^2y^3}}_{\frac{\partial}{\partial x}\log(1+x^2y^3)}\,dx + \underbrace{\frac{2kx^2y^2}{1+x^2y^3}}_{\overset{?}{=}\,\frac{\partial}{\partial y}\log(1+x^2y^3)}\,dy$

$$\frac{\overset{\shortparallel}{3x^2y^2}}{1+x^2y^3}$$

Uguali per $2k=3$, cioè
$k=3/2$

$\boxed{18}$ (2) Trovare tutti i $v \in \mathbb{C}^3$ unitari, $v_1 \in i\cdot\mathbb{R}$
$v_1+v_2+v_3=0$, $v \perp (1-i)e_1+2ie_2+(3+i)e_3$.

•) $$\left\langle \begin{pmatrix} i \\ z \\ -i-z \end{pmatrix} \middle| \begin{pmatrix} 1-i \\ 2i \\ 3+i \end{pmatrix} \right\rangle = 0$$

$$i(1+i)+z\cdot(-2i)+(-1-z)\cdot(3-i)=0$$

risolvo rispetto a z; poi $\pm$ normalizzo

•) $v_1+v_2+v_3=0 \iff v \perp \begin{pmatrix}1\\1\\1\end{pmatrix}$.

Trovo vettore ortog. a $\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1-i\\2i\\3+i\end{pmatrix}$;

poi moltiplico per numero in modo che v_1

diventi immaginario; infine $\pm$ normalizzo.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \wedge \begin{pmatrix}1+i\\-2i\\3+i\end{pmatrix} = \begin{pmatrix}3+i\\-2+2i\\-1-3i\end{pmatrix}.$$

Ora moltiplico per $i(3-i) = 1+3i$ _

$$(1+3i)\begin{pmatrix} 3+i \\ -2+2i \\ -1-3i \end{pmatrix} = \begin{pmatrix} 10i \\ -8-4i \\ 8-6i \end{pmatrix} = 2\begin{pmatrix} 5i \\ -4-2i \\ 4-3i \end{pmatrix}$$

Conclusione $\pm \dfrac{1}{70}\begin{pmatrix} 5i \\ -4-2i \\ 4-3i \end{pmatrix}$ _

$\boxed{18}$ 6 Esibire la matrice hessiana di

$f(x,y) = 3\log(1+2xy^4)$ in $(1,-1)$

e segni autovalori _

$$\frac{\partial f}{\partial x} = \frac{9 \cdot 2\, y^4}{1+2xy^4} \qquad \frac{\partial f}{\partial y} = \frac{9 \cdot 8\, xy^3}{1+2xy^4}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{9 \cdot 2 \cdot y^4 \cdot 2y^4}{(1+2xy^4)^2} \text{ ; in } (1,-1) \text{ fa } -4$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{9 \cdot 2 \cdot 4 \cdot y^3 (1+2xy^4) - 9 \cdot 2 \cdot y^4 \cdot 8xy^3}{(1+2xy^4)^2} \longrightarrow \frac{9 \cdot 8 \cdot (-3) - 9 \cdot 2 \cdot (-8)}{9} = -8$$

$$\frac{\partial^2 f}{\partial y^2} = \dots \longrightarrow 8$$

$$Hf = \begin{pmatrix} -4 & -8 \\ -8 & 8 \end{pmatrix}$$

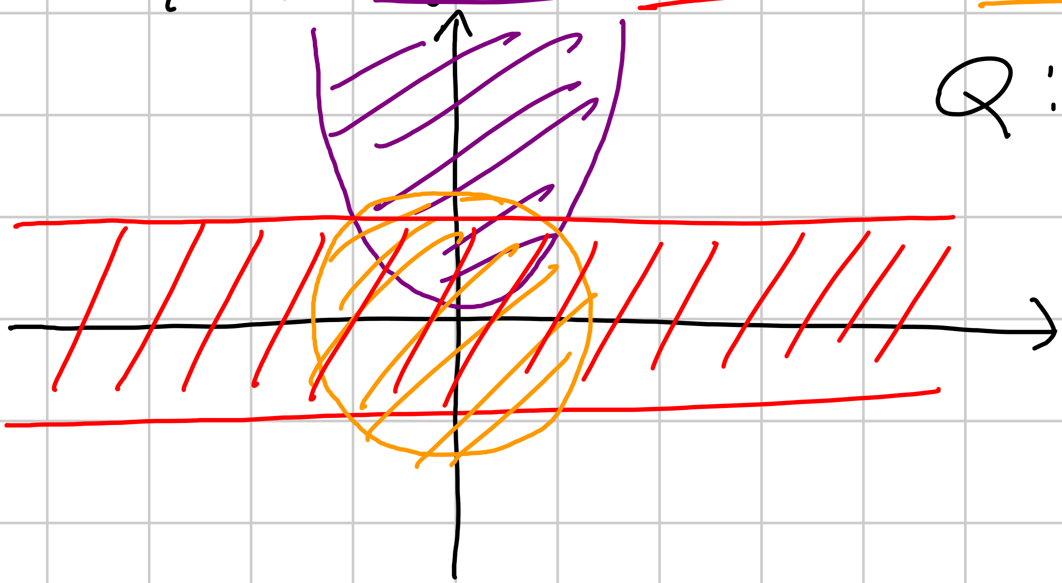
$\det < 0$

$\Longrightarrow$ autoval $+, -$

$\boxed{18}$ 7 Esistono forme chiuse non esatte su

$$\Omega = \{ (x,y) : \ y > x^2 \ \text{o} \ |y| < 1 \ \text{o} \ x^2 + y^2 < 2 \}$$

Q: Ω ha buchi?



NO

$\boxed{20}$ (1) Trovare tutti i vettori di $\mathbb{R}^3$ unitari che formano angolo $\pi/3$ con e_1+e_2 e con e_2-e_3

$\cos \pi/3 = 1/2$

$$\begin{cases} x^2+y^2+z^2=1 \\ \dfrac{x+y}{\sqrt{x^2+y^2+z^2}\cdot\sqrt{2}} = \dfrac{1}{2} \\ \dfrac{y-z}{\sqrt{\dots}\cdot\sqrt{2}} = \dfrac{1}{2} \end{cases}$$

$$\begin{cases} x^2+y^2+z^2=1 \\ x = \dfrac{1}{\sqrt{2}}-y \\ z = y-\dfrac{1}{\sqrt{2}} \end{cases}$$

$$\left(\frac{1}{\sqrt{2}}-y\right)^2+y^2+\left(y-\frac{1}{\sqrt{2}}\right)^2=1$$

$$\cancel{\frac{1}{2}} - \sqrt{2}\,y + y^2 + y^2 + y^2 - \sqrt{2}\,y + \cancel{\frac{1}{2}} = \cancel{1}$$

$$y(3y - 2\sqrt{2}) = 0 \qquad y = 0 \qquad y = \frac{2}{3}\sqrt{2}$$

$$\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \qquad + \frac{1}{3\sqrt{2}}(-1, 4, 1)$$

$\boxed{21}$ 2 Per quali k la $\begin{pmatrix} 1 & 0 & 0 \\ 3 & k & 0 \\ k-2 & 2 & k^2 \end{pmatrix}$ non è diago?

Matrice triang (sup) $\Longrightarrow$ autoval sono i coeff sulla diagonale principale

$\lambda_1 = 1 \quad \lambda_2 = k \quad \lambda_3 = k^2$

Se sono distinti, cioè per $k \neq 0, +1, -1$

certamente è diago.

$k = 0$, autoval 0 doppio $\quad A - 0 \cdot I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ -2 & 2 & 0 \end{pmatrix}$

$\text{rank} = 2 \Rightarrow m.g. = 3 - 2 = 1$ non diago

$k = 1$, autoval 1 triplo ma non è già diagonale

$\Rightarrow$ non diagonalizzabile

$k=-1$ autoval 1 doppio $\quad$ (-1 singolo: no prob)

$$m.g.(+1) = 3 - \text{rank}\left(A - 1\cdot I_3\right) =$$

$$= 3 - \text{rank}\begin{pmatrix} 0 & 0 & 0 \\ 3 & -2 & 0 \\ -3 & 2 & 0 \end{pmatrix} = 3-1 = 2$$

è diago.

Conclusione: non diago per $k=0$ e $k=1$.

$\boxed{21}$ (4) Per quali $k \in \mathbb{R}$ esiste base ortonormale di $\mathbb{R}^2$ costituita da autovettori di

$$A_k = \begin{pmatrix} k^3 & k^2-3 \\ 2k & 1-k \end{pmatrix}$$

$\Leftrightarrow$

per quali k che A_k è simmetrica?

$k^2-3=2k \qquad k^2-2k-3=0 \qquad (k-3)(k+1)=0$

$k=3$ o $k=-1$.

$\boxed{39}$ (tutto)

(1) Trovare autoval di $\begin{pmatrix} 2+i & 4i-3 \\ i & -1 \end{pmatrix}$

e base di $\mathbb{C}^2$ che la diagonalizza.

$$tr = 1+i \qquad = \lambda_1 + \lambda_2$$

$$\det = -2-i+4+3i = 2+2i \qquad = \lambda_1 \cdot \lambda_2$$

$$\Delta = (1+i)^2 - 4(2+2i) = 2i - 8 - 8i = -8-6i$$

$$\sqrt{\Delta} = \pm (a+ib) \qquad \begin{cases} a^2 - b^2 = -8 \\ ab = -3 \end{cases}$$

$$\Rightarrow \sqrt{\Delta} = \pm (1-3i)$$

$$\lambda_{1,2} = \frac{1+i \pm (1-3i)}{2} = \begin{cases} 1-i \\ 2i \end{cases}$$

$$\lambda_1 = 1-i \ : \ \begin{cases} \cdots \\ ix - y = (1-i)y \end{cases} \qquad v_1 = \begin{pmatrix} 2-i \\ i \end{pmatrix}$$

(o suoi multipli)

$$\lambda_2 = 2i \ : \ \begin{cases} \cdots \\ ix - y = 2iy \end{cases} \qquad v_2 = \begin{pmatrix} 1+2i \\ i \end{pmatrix}$$

(o suoi multipli)

② Trovare i $v \in \mathbb{C}^2$ unitari, $\perp \begin{pmatrix} 1+2i \\ 3-i \end{pmatrix}$, $v_1 \in i \cdot \mathbb{R}$.

$$\left\langle \begin{pmatrix} i \\ z \end{pmatrix} \middle| \begin{pmatrix} 1+2i \\ 3-i \end{pmatrix} \right\rangle = 0$$

$$i(1-2i) + z(3+i) = 0 \qquad z = -\frac{7+i}{10}$$

$$v = \pm \frac{1}{5\sqrt{6}} \begin{pmatrix} -10i \\ 7+i \end{pmatrix}$$

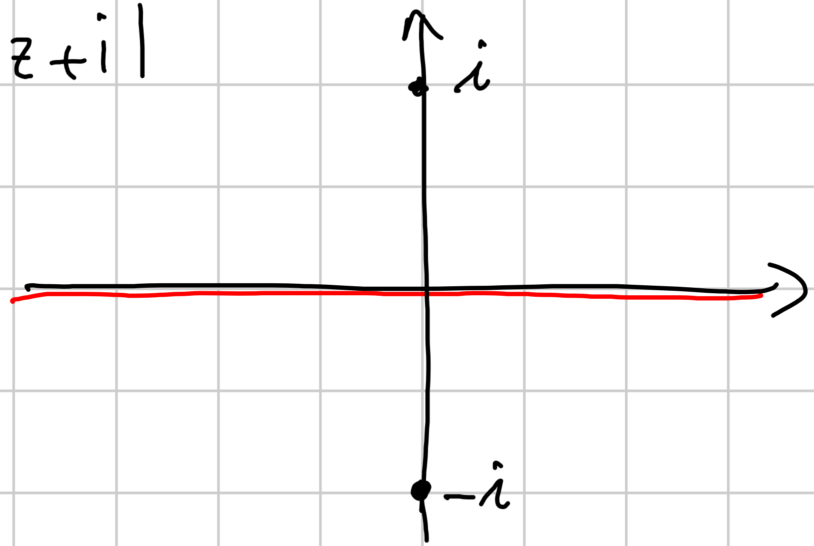
③ Per quali $z \in \mathbb{C}$ la matrice $\begin{pmatrix} 1+z & z-i \\ z+i & z \end{pmatrix}$ ammette base ortonorm. di autovettori?

Equivale a: per quali z è normale?

$$\begin{pmatrix} 1+z & z-i \\ z+i & z \end{pmatrix}\begin{pmatrix} 1+\bar z & \bar z-i \\ \bar z+i & \bar z \end{pmatrix} = \begin{pmatrix} 1+\bar z & \bar z-i \\ \bar z+i & \bar z \end{pmatrix}\begin{pmatrix} 1+z & z-i \\ z+i & z \end{pmatrix}$$

$$\begin{pmatrix} |1+z|^2+|z-i|^2 & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} |1+z|^2+|z+i|^2 & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\begin{cases} |z-i| = |z+i| \\ \text{—} \\ \text{—} \end{cases}$$



$$\begin{cases} z \in \mathbb{R} \\ \text{—} \\ \text{—} \\ \text{—} \end{cases}$$

Per $z \in \mathbb{R}$ la matrice hermitiana $\Rightarrow$ D normale -

Conclusione: $z \in \mathbb{R}$ -

(4) $C = \{ [x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{R}) \;;\; x_0^2 = x_1 x_2 \}$

$E_j = \{ [x_0 : x_1 : x_2] : \quad x_j = 1 \}$

Tipo affine di $E_j \cap C_1$:

$E_0 \cap C_1$: $\quad x_1 x_2 = 1$ iperbole

$E_1 \cap C_1$: $\quad x_0^2 = x_2$ parabole

$E_2 \cap C_1$: $\quad x_0^2 = x_1 \quad$ parabola

⑤ Tipo affine di $\quad x^2 - y^2 - 4xy + 4xz + 6yz + 2y + 2z + 1 = 0$

$$\begin{pmatrix} 1 & -2 & 2 & 0 \\ -2 & 1 & 3 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$(d_1 > 0) \quad d_2 < 0$

$d_3 = -8 - 8 - 4 - 9 < 0$

$$d_4 = \det \begin{pmatrix} 1 & -2 & 2 & 0 \\ -2 & 0 & 2 & 0 \\ 2 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \dots < 0$$

Segni autoval $(+ - +)+$

$$x^2 - y^2 + z^2 + 1 = 0$$

$Z^2 = 1 + X^2 + Y^2$ iperb. a 2 falde

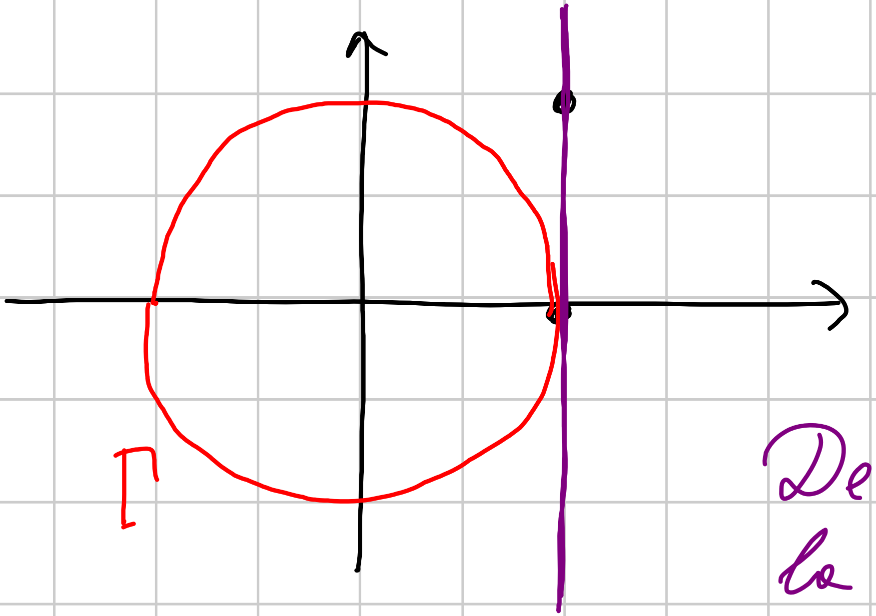
(ellittico)

(6)

$$\Gamma = \{(x_0, x_1) : x_0^2 + x_1^2 = 1\} \subset \mathbb{R}^2 \subset \mathbb{P}^2(\mathbb{R})$$

Trovare $f : \mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ t.c.

$f(\Gamma) \cap \mathbb{R}^2$ sia una parabola.



Cioè: cerco f t.c. $f(\Gamma)$ abbia un solo punto all'∞.

Decido di mandare all'∞ la retta viola:

$$f : \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$f : \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f: \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow f = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

⑦ Calcolare $\displaystyle\int_\alpha (2xy^3 dx + 3x^2y^2 dy)$ $\quad \alpha(t) = \begin{pmatrix} t \cdot \log_2(1+t) \\ -\sin(\pi t/2) \end{pmatrix}$

$t \in [0,1]$

$$= \int_\alpha d(x^2y^3) = x^2y^3 \Big|_{\alpha(0)}^{\alpha(1)}$$

$$= x^2 y^3 \Big|_{(0,0)}^{(1,-1)} = 1^2 \cdot (-1)^3 = -1$$

$\boxed{13}$ (1) Trovare punto di $X = \{x \in \mathbb{R}^3 : 3x_1 - x_2 + 4x_3 = 0\}$ avente minima distanza da $v = -e_1 + 5e_2 + 4e_3$.

Cioè: trovare la proiez. ortog. di v su X.

$X = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}^{\perp}$ cioè $X^{\perp} = \mathrm{Span}\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

$$p_X(v) = v - p_{X^\perp}(v) =$$

$$= \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} - \frac{-3-5+16}{9+1+16} \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} - \frac{4}{13} \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} -25 \\ 69 \\ 36 \end{pmatrix}.$$

(2) Per quali $z \in \mathbb{C}$ la $\begin{pmatrix} 0 & z^2 \\ -9 & z-3 \end{pmatrix}$ ha base ortonormale di autovettori e autovalori $\in i\mathbb{R}$?

Equivale a: per quali z le $\begin{pmatrix} 0 & z^2 \\ -9 & z-3 \end{pmatrix}$ è antihermitiana?

$$\begin{cases} 0 \in i\mathbb{R} \quad \checkmark \\ z-3 \in i\mathbb{R} \\ z^2 = -\overline{(-9)} \end{cases} \qquad \begin{cases} \checkmark \\ z = 3 + it \\ z = \pm 3 \end{cases}$$

$$z = 3$$

③ Trovare base ortonormale di $\mathbb{R}^2$ che diagonalizza $\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$ -

ATT: vedo che è simm $\Longrightarrow$ qualsiasi base di autovett. è ortogonale. Dunque devo:

- trovare una base di autovett
- normalizzare

NON: ortonormalizzare (inutile)

$$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$$

$tr = 2$

$det = -17$

$$\lambda_{1,2} = 1 \pm \sqrt{1+17} = 1 \pm 3\sqrt{2}$$

$\lambda_1 = 1+3\sqrt{2}$;
$$\begin{cases} -2x+3y=(1+3\sqrt{2})x \\ \text{—} \end{cases}$$
$\hat{v}_1 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$

$$v_1 = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

$\lambda_2 = 1-3\sqrt{2}$;
$$\begin{cases} -2x+3y=(1-3\sqrt{2})x \\ \text{—} \end{cases}$$

$$v_2 = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

④ Trovare

$$\{[1+t : 2-t : t-6] : t \in \mathbb{R}\} \cap \left(\underbrace{\begin{matrix} \text{Punti a } \infty \text{ di} \\ x^2+xy-4yz+7y=5 \end{matrix}}_{\{[x:y:z:0] \,:\, x^2+xy-4yz=0\}} \right)$$

$$(1+t)^2+(1+t)(2-t)-4(2-t)(t-6)=0$$

(cont)

$$4t^2-29t+51=0$$

$$\Delta = 841 - 16(50+1) = 841 - 800 - 16 = 25$$

$$t_{1,2} = \frac{29 \pm 5}{8} = \begin{cases} \frac{17}{4} \\ 3 \end{cases}$$

$$(4t-17)(t-3)$$

$t = \frac{17}{4}$: $\left[1 + \frac{17}{4} : 2 - \frac{17}{4} : \frac{17}{4} - 6\right] = [21 : -9 : -7]$

$= [-21 : 9 : 7]$

$t = 3$: $[4 : -1 : -3] = [-4 : 1 : 3]$

⑤ Per quali $a \in \mathbb{R}$ l'eq.

$$\underbrace{(x+2y-z)}_{X}^2 - a\,\underbrace{(3x-y+1)}_{Y}^2 = (a+1)\underbrace{(y-2z)}_{Z}$$ è parab. ell.?

È buon cambio di variabili? Sì se det $\begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \\ -1 & 1 & -2 \end{pmatrix} \neq 0$

VERO

Dunque la quadrica è

$$(a+1)\cdot Z = X^2 - aY^2 \; .$$

Eq. can. parab. ell : $\quad u = w^2 + v^2$

Devo avere $\quad a < 0 \quad$ con $\quad a \neq -1$.

⑥ Trovare $\{[1+t : 5-t : -2] : t \in \mathbb{R}\} \cap \{[1+4t : -6 : 4+t] : t \in \mathbb{R}\}$

Impongo rank $\begin{pmatrix} 1+t & 1+4s \\ 5-t & -6 \\ -2 & 4+s \end{pmatrix} \leq 1$ cioè tutti i det 2×2 nulli.

$(5-t)(4+s) - 12 = 0 \qquad s = \dfrac{4t-8}{5-t}$ (II+III riga)

$(1+t)(4+s) + 2(1+4s) = 0$ (I+III riga)

$(1+t)\dfrac{\cancel{12}^{\,6}}{5-t} + \cancel{2}^{\,1}\left(1 + \dfrac{16t-32}{5-t}\right) = 0$

$6+6t+5-t+16t-32=0$ $\qquad 21t=21 \qquad t=1$

$s=-1$

$$[1+t : 5-t : -2] \xrightarrow{t=1} [2 : 4 : -2]$$

$$[1+4s : -6 : 4+s] \xrightarrow{s=-1} [-3 : -6 : 3]$$

Sono uguali: ok

$[1 : 2 : -1]$

(7) $\displaystyle\int_\alpha \frac{dx+dy}{x+y}$ $\quad \alpha(t) = \begin{pmatrix} 3\sin^2(2t) \\ 2\cos^2(3t) \end{pmatrix}$ $\quad t \in [-\frac{\pi}{4}, 0]$

$$= \int_\alpha d(\log(x+y)) = \log(x+y)\Big|_{\alpha(-\pi/4)}^{\alpha(0)}$$

$$= \log(x+y)\Big|_{(3,1)}^{(0,2)} = \log(0+2) - \log(3+1)$$

$$= \log 2 - \log 4 = -\log 2$$