Geometria 27/3/14

(10) $A \in M_{n \times n}(\mathbb{R}) \qquad b_{ij} = (-1)^{i+j} \cdot a_{ij}$

$\langle \cdot | \cdot \rangle_A$ prod scal $\Longleftrightarrow$ $\langle \cdot | \cdot \rangle_B$ prod. scal.

$$\langle x|y \rangle_B = {}^t x \cdot B \cdot y = (x_1 \dots x_n) \begin{pmatrix} b_{11} & \dots & \\ \vdots & & \\ \vdots & & \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \sum_{i,j=1}^{n} x_i \cdot b_{ij} \cdot y_j = \sum_{i,j=1}^{n} x_i \cdot (-1)^{i+j} \cdot a_{ij} \cdot y_j$$

$$= \sum_{i,j=1}^{n} \left((-1)^i \cdot x_i \right) \cdot a_{ij} \cdot \left((-1)^j \cdot y_j \right)$$

$$= \langle f(x) | f(y) \rangle_A \qquad \left(f(z) \right)_k = (-1)^k \cdot x_k$$

f lineare invertibile — $\underline{\text{OK}}$

(11) $v_1, \dots, v_n \in \mathbb{R}^n \qquad a_{ij} = \langle v_i / v_j \rangle_{\mathbb{R}^n}$

$\langle \cdot | \cdot \rangle_A$ prod. scal?

bil ✓

simm ✓

Posto $X = (v_1, \dots, v_n)$ abbiamo $A = {}^tX \cdot X$

$$\Rightarrow \langle x | x \rangle_A = {}^t x \cdot A \cdot x = \underbrace{{}^t x \cdot {}^t X}_{{}^t(X \cdot x)} \cdot X \cdot x$$

$$= \| X \cdot x \|^2$$

Sempre ≥ 0; è > 0 $\forall x \neq 0$

$\Longleftrightarrow X$ invertibile $\Longleftrightarrow v_1, \dots, v_n$ base di $\mathbb{R}^4$ _

(12) Data $f: V \times V \to \mathbb{R}$ bil. simm poniamo $g(v) = f(v,v)$ — Trovare f sapendo g:

(a) $V = \mathbb{R}^2$ $\quad q(x) = 3x_1^2 + 2x_1x_2 - x_2^2 = \langle x | x \rangle$

$$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$$

$$f(x,y) = \langle x | y \rangle_A =$$

$$= 3x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$$

(b) $\mathbb{R}^3$ $\quad q(x) = -2x_1^2 + x_2^2 - 5x_3^2 - x_1x_2 + 3x_1x_3 + 2x_2x_3$

$$= \langle x | x \rangle \begin{pmatrix} -2 & -1/2 & 3/2 \\ -1/2 & 1 & 1 \\ 3/2 & 1 & -5 \end{pmatrix}$$

$$f(x,y) = -2x_1y_1 + x_2y_2 - 5x_3y_3 - \frac{1}{2}(x_1y_2 + x_2y_1) + \frac{3}{2}(x_1y_3 + x_3y_1) + x_2y_3 + x_3y_2$$

(c) $V = M_{2\times 2}$ $\quad q(A) = \det(A)$

$$q\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$f(A,B) = f\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right)$$

$$= \frac{1}{2}\left(a_{11} b_{22} + a_{22} b_{11} - a_{21} b_{12} - a_{12} b_{21} \right)$$

(d) $V = M_{3\times 3}$ $\qquad g(A) = \sum_{j=1}^{2} (A^2)_{j,3-j}$

$$g(A) = (A^2)_{1,2} + (A^2)_{2,1}$$

$$= \underline{a_{11} a_{12}} + \underline{a_{12} a_{22}} + a_{13} a_{32}$$

$$+ a_{21} a_{11} + a_{22} a_{21} + a_{23} a_{31}$$

$$f(A,B) = \frac{1}{2}\left(\underline{a_{11} b_{12} + a_{12} b_{11}} + \underline{a_{12} b_{22} + a_{22} b_{12}} + \ldots \right)$$

(e) $V = \mathbb{R}_{\leq 2}[t]$ $\quad q(p(t)) = p(1) \cdot p(-2)$

$$f(p(t), r(t)) = \frac{1}{2}\left(p(1) \cdot r(-2) + p(-2) \cdot r(1) \right)$$

13 $\mathbb{R}_{\leq 2}[t]$ $\quad \langle p(t) | q(t) \rangle = p(0) \cdot q(0) + p(1) \cdot q(1) + p(2) \cdot q(2)$

(Prod. scal: bil. ✓ simm ✓ ; def. pos:

$$\langle p(t) | p(t) \rangle = p(0)^2 + p(1)^2 + p(2)^2$$

sempre ≥ 0; nullo solo se $p(0) = p(1) = p(2) = 0$
ovvero $p(t) = 0$, poiché $\deg(p(t)) \leq 2$.)

Cercare $p(t)$ ortog a $1+t$ e $1+t^2$ con $\| \| = \sqrt{5}$

Cerco $p(t) = a_0 + a_1 t + a_2 t^2$

$$\begin{cases} p(t) \perp (1+t) \\ p(t) \perp (1+t^2) \end{cases} \qquad \begin{cases} \langle p(t) \mid 1+t \rangle = 0 \\ \langle p(t) \mid 1+t^2 \rangle = 0 \end{cases}$$

$$\begin{cases} a_0 \cdot 1 + (a_0 + a_1 + a_2) \cdot 2 + (a_0 + 2a_1 + 4a_2) \cdot 3 = 0 \\ a_0 \cdot 1 \quad (a_0 + a_1 + a_2) \cdot 2 \quad (a_0 + 2a_1 + 4a_2) \cdot 5 = 0 \end{cases}$$

$$\begin{cases} 3a_0 + 2a_1 + 2a_2 = 0 \\ a_0 + 2a_1 + 4a_2 = 0 \end{cases} \qquad \begin{cases} a_0 = -2a_1 - 4a_2 \\ -4a_1 - 10a_2 = 0 \end{cases}$$

$$\begin{cases} a_1 = -5 \\ a_2 = 2 \\ a_0 = 2 \end{cases} \qquad \tilde{p}(t) = 2 - 5t + 2t^2$$

$$\|\tilde{p}(t)\|^2 = \tilde{p}(0)^2 + \tilde{p}(1)^2 + \tilde{p}(2)^2$$
$$= 4 + 1 + 0 = 5$$

$\Rightarrow$ $2 - 5t + 2t^2$ è o.k. _

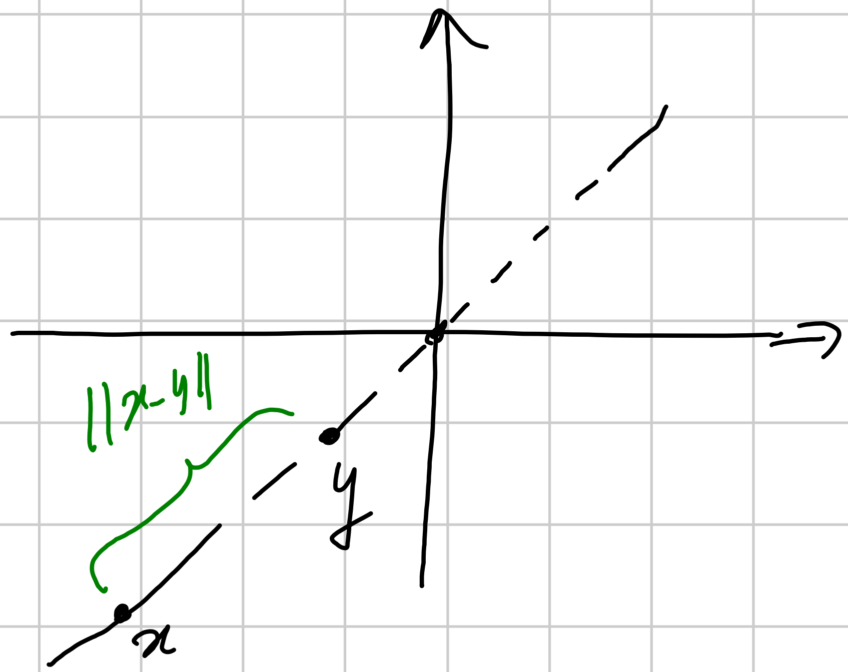
(14) $d : V \times V \to \mathbb{R}$ distanza se

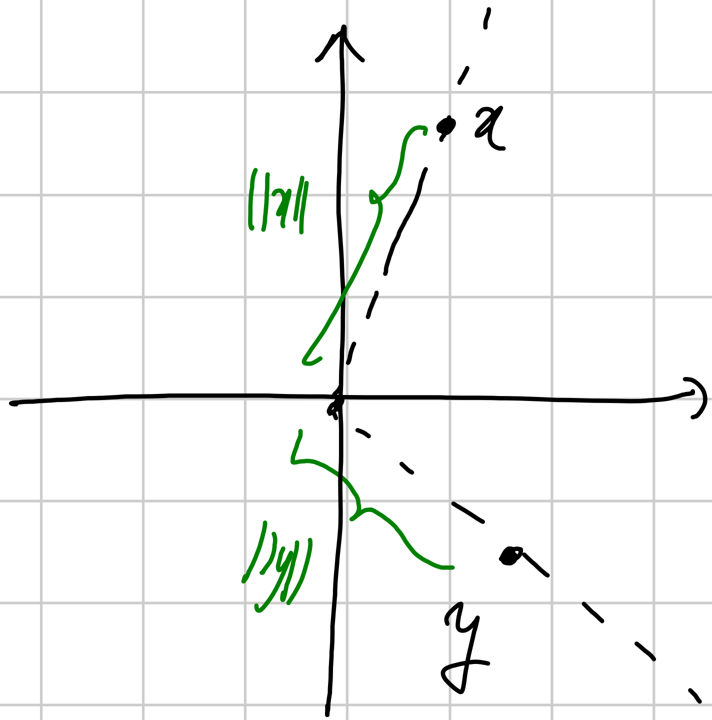
•) $d(v,w) > 0$ per $v \neq w$, $d(v,v) = 0$

•) $d(v,w) = d(w,v)$

•) $d(v,w) \leq d(v,z) + d(z,w)$ _

$$
\mathrm{SNFC}(x,y) = \begin{cases} \|x-y\| & \text{se lin. dip.} \\ \|x\| + \|y\| & \text{altrimenti.} \end{cases}
$$





(chiaro che è distanza) ; dimo formule ;

•) pos ✓ •) simm ✓ •) dis. tria.

$SNCF(x,y) \leq SNCF(x,z) + SNCF(z,y)$ Casi :

• x, y, z tutti lin. dip. $\|x-y\| \leq \|x-z\| + \|z-y\|$ ✓

• x, y lin. dip., z no $\|x-y\| \leq \|x\| + \|z\| + \|z\| + \|y\|$ ✓

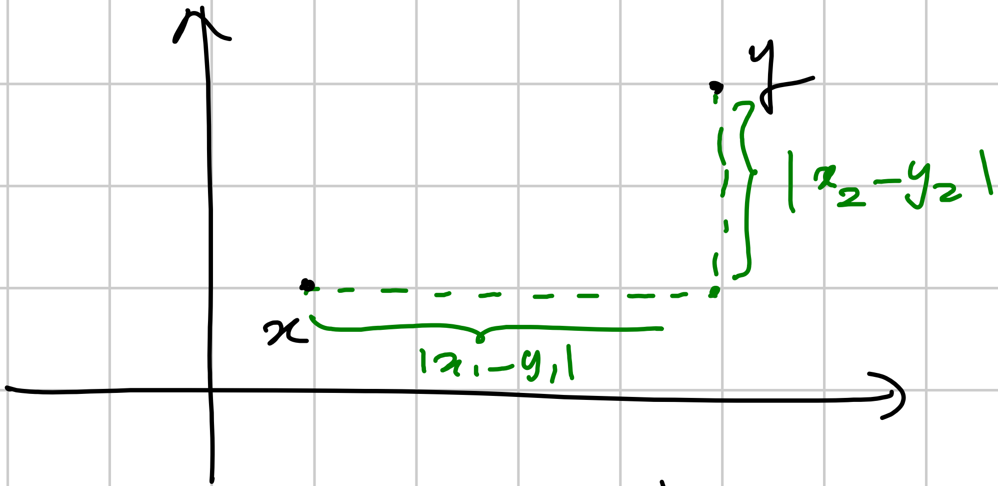
- x, z lin. dip. y no $\quad \|x\| + \|y\| \leq \|x - z\| + \|z\| + \|y\|$

equivale a

$\|(x - z) + z\| \leq \|x - z\| + \|z\|$ ✓

- y, z dip. x no : analogo al prec.

$$NYC(x, y) = \sum |x_i - y_i|$$



(chiaro che è distanza) : verifica formula

•) pos ✓ •) simm ✓

•) $NYC(x,y) \le NYC(x,z) + NYC(z,y)$

$$\sum |x_i - y_i| \le \sum |x_i - z_i| + \sum |z_i - y_i| \quad ✓$$

15 ortonormalizzare

(a) $V = \mathbb{R}^2$, $\langle \cdot | \cdot \rangle_{\mathbb{R}^2}$ $\quad \begin{pmatrix} 5 \\ -12 \end{pmatrix} \quad \begin{pmatrix} \sqrt{\pi} \\ -1789 \end{pmatrix}$

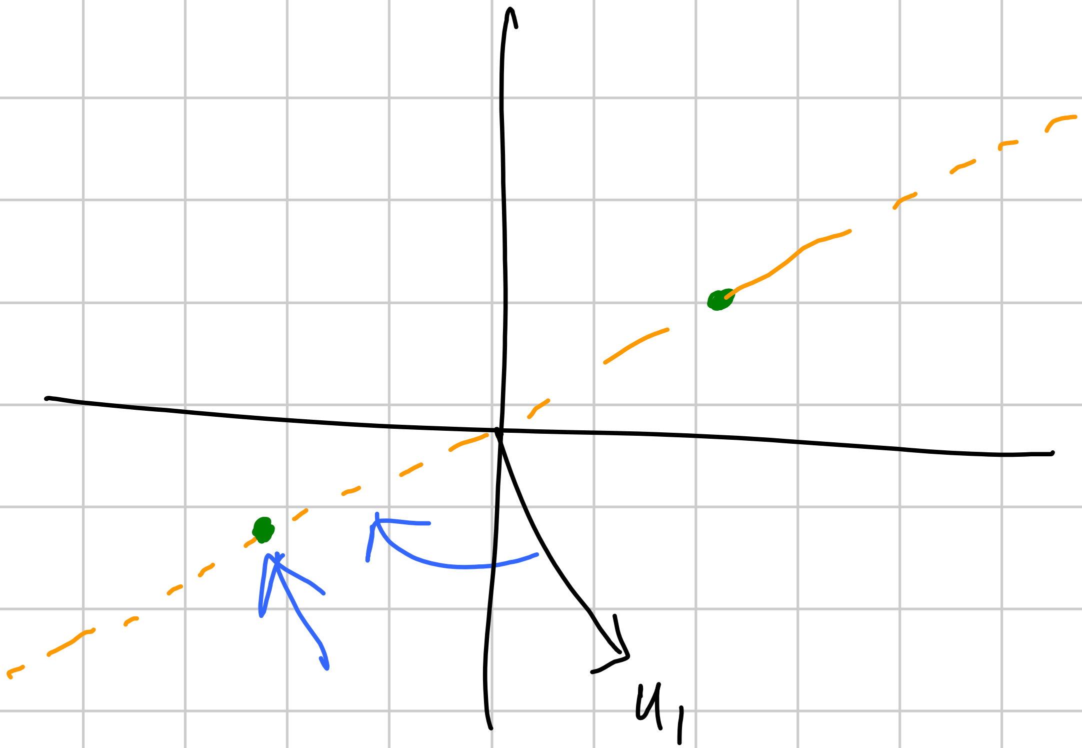
$$u_1 = \frac{1}{13} \begin{pmatrix} 5 \\ -12 \end{pmatrix}$$

$$\tilde{u}_2 = \begin{pmatrix} \sqrt{\pi} \\ -1789 \end{pmatrix} - \frac{5\sqrt{\pi} + 12 \cdot 1789}{165} \begin{pmatrix} 5 \\ -12 \end{pmatrix}$$

$$u_2 = \tilde{u}_2 / \| \tilde{u}_2 \|$$

No

Invece: sappiamo che $u_2 \perp u_1$, $\|u_2\|$,



$\det(u_1, u_2)$ concorde con

$\det\begin{pmatrix} 5 & \sqrt{\pi} \\ -12 & -1789 \end{pmatrix}$

$\Rightarrow$ negativo

$\Rightarrow u_2 = -\frac{1}{13}\begin{pmatrix} 12 \\ 5 \end{pmatrix}$