ETA - 23/10/13

<u>Teo</u>: Immersioni di S^1 in $\mathbb{R}^2 /_{\text{omotopia reg.}} \stackrel{\deg}{\longleftrightarrow} \mathbb{Z}$

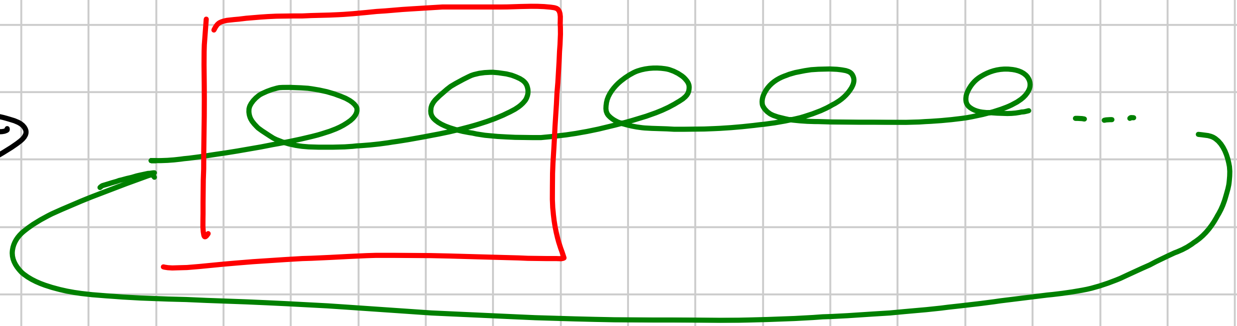
<u>Cor</u>: Immersioni di S^1 in $S^2 /_{\text{omotopia reg.}} \longleftrightarrow \mathbb{Z}/2$

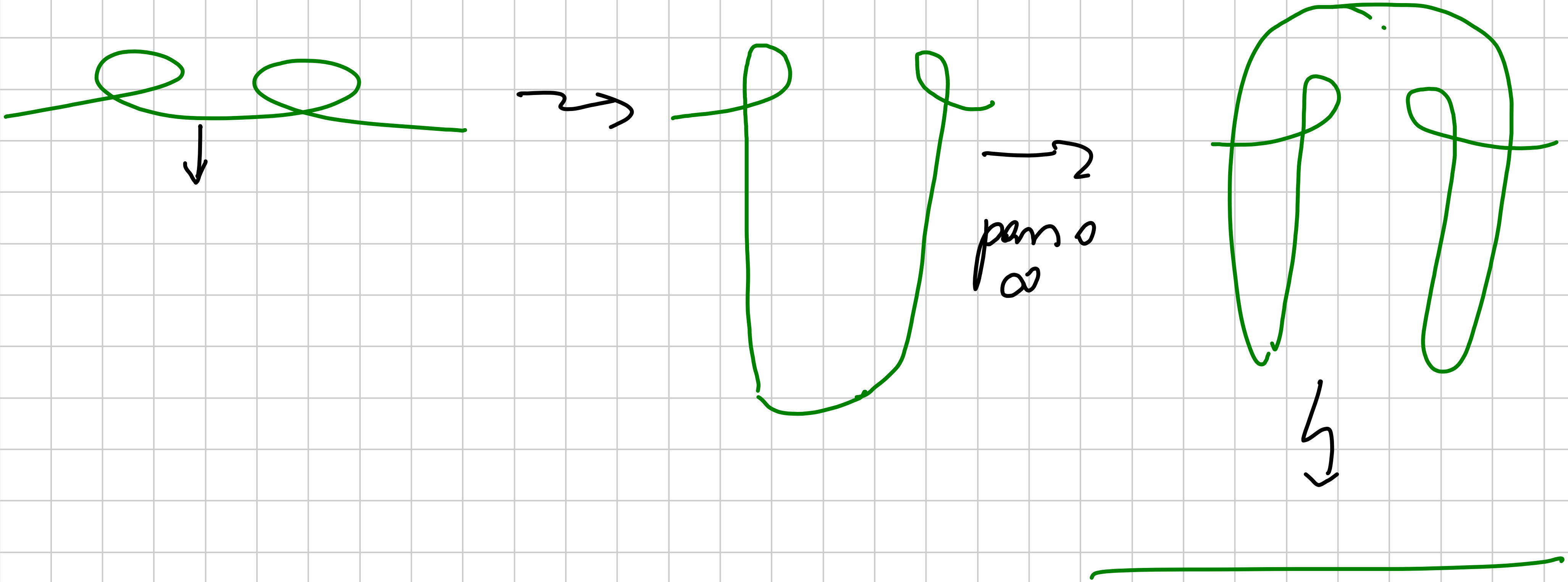
<u>Dim</u>: sia $f : S^1 \longrightarrow S^2 = \mathbb{R}^2 \cup \{\infty\}$

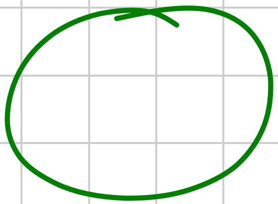
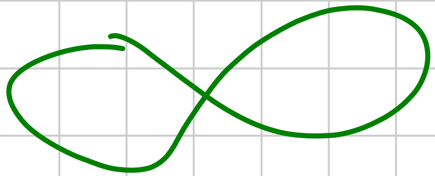
Wlog $\infty \notin \mathrm{Im}(f) \implies \deg \frac{f'}{\|f'\|} =: d$

provo che posso ridurmi (via omotop. reg. su $\int^{1 2}$) a $d=0$ o $d=1$.

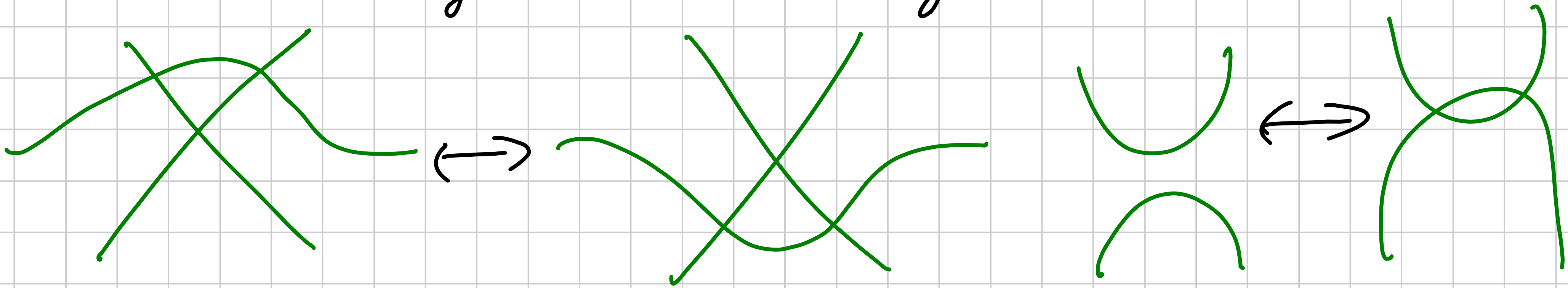
omotop. reg. in $\mathbb{R}^2 \rightsquigarrow$



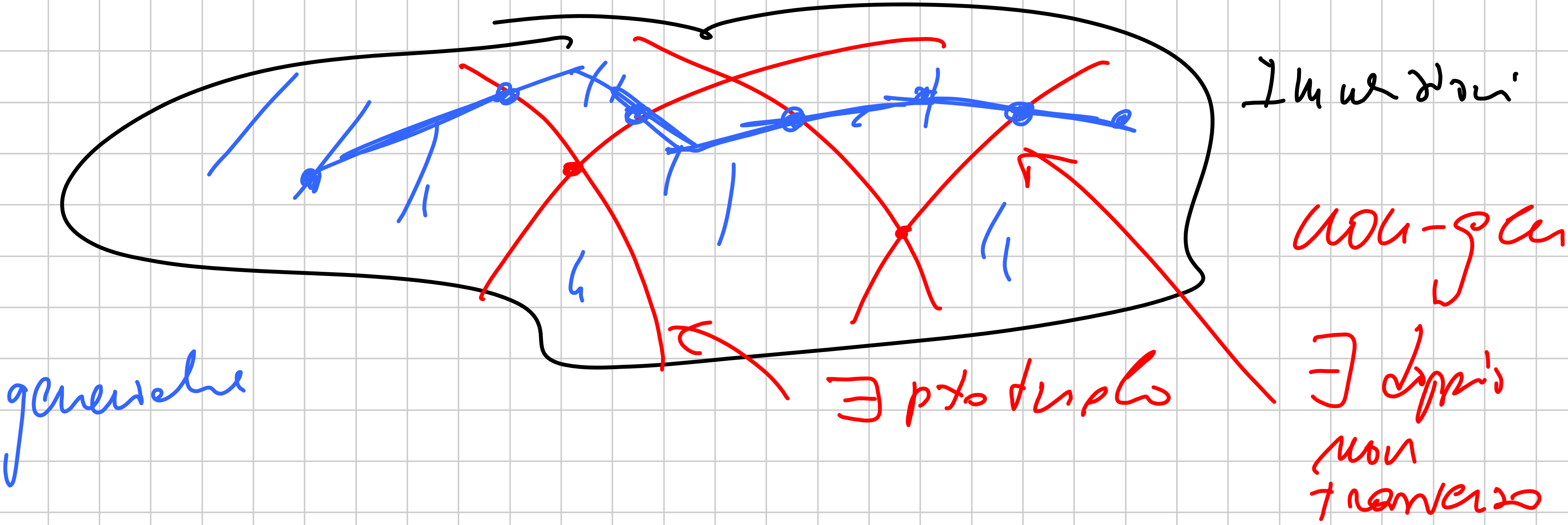


Resta da vedere che  ≠  !

segue dal fatto che l'omotopia regolare tra immersioni generiche è generata da:



Immersioni



Le mosse non cambiano la parità del numero di pti doppi.

Teo: $\mathbb{R}^n$ non è omeo a $\mathbb{R}^m$ per $n \neq m$.

Dim: p.a. $\mathbb{R}^n \cong \mathbb{R}^m$ $\quad n > m$

$$\Longrightarrow \begin{array}{ccc} \mathbb{R}^n \setminus \{pto\} & \cong & \mathbb{R}^m \setminus \{pto\} \\ \wr\!\wr & & \wr\!\wr \\ S^{n-1} \cong \partial\Delta_n & & S^{m-1} \cong \partial\Delta_m \end{array}$$

ma $H_{n-1}(S^{n-1}) = \mathbb{Z}$, $H_{n-1}(S^{m-1}) = 0$. $\blacksquare$

Teo (Jordan - Schönflies) :

$\gamma \subset \mathbb{R}^2$, $\gamma \cong S^1$ (omeo)

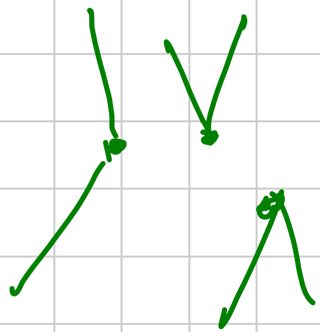
$\Rightarrow$ $\mathbb{R}^2 \setminus \gamma = D \cup E$, $\gamma = \partial D = \partial E$,

$\overline{D} \cong D^2$, $\overline{E} \cong \mathbb{R}^2 \setminus \mathring{D}^2$. (DURO)

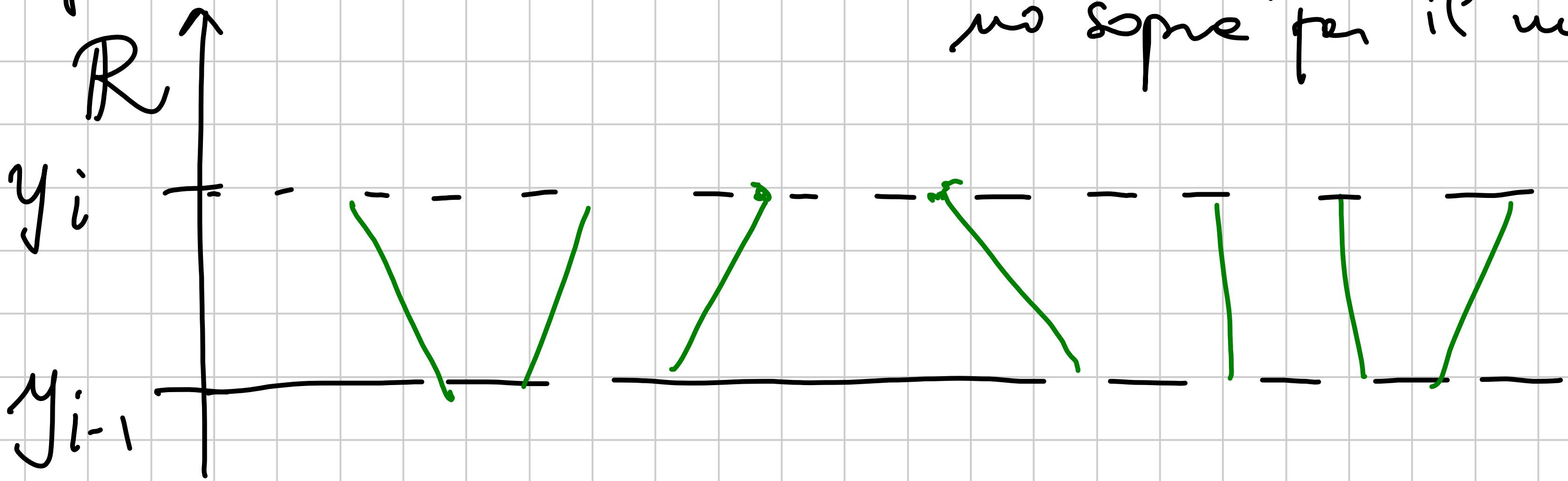
Oss: per $S^2 \subset \mathbb{R}^3$ è falso (versione TOP).

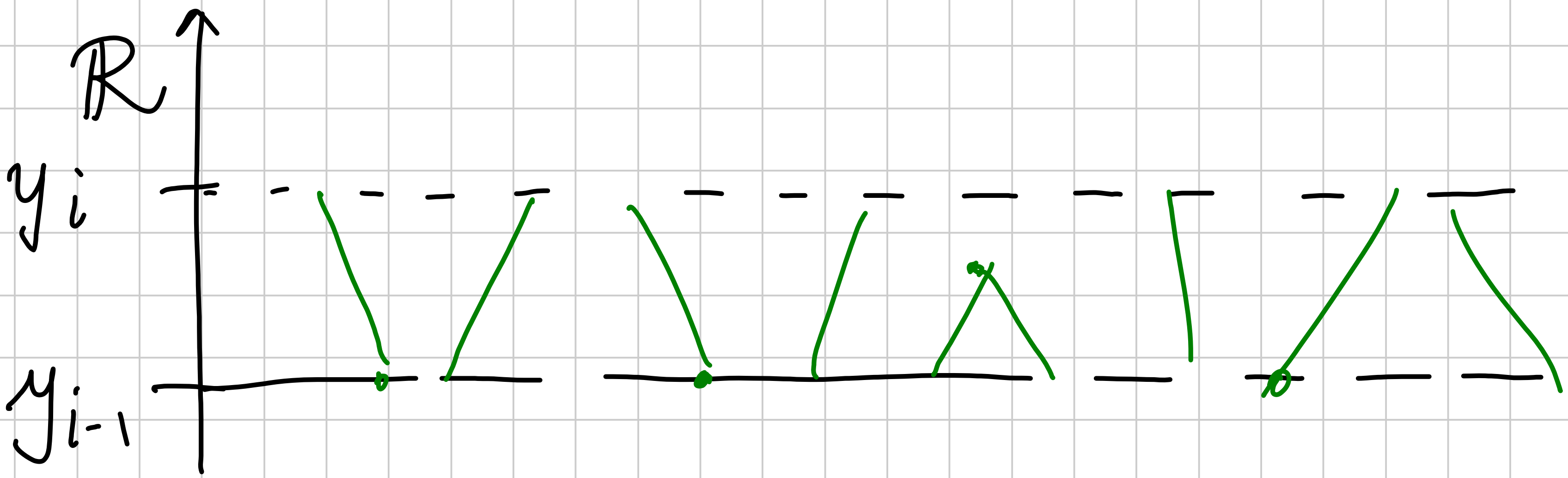
<u>Dimo</u> (PL / Diff) : $K \subset \mathbb{R}^2$ complesso simpliciale
poi
$|K| \cong_{PL} S^1 \ (\partial \Delta_2)$.

Wlog / rotaz. suppongo che nessun $e \in K^{[1]}$ sia orizzontale e che tutti i vertici siano ad ordinate distinte. Scelgo altezze $y_1 < y_2 < \dots < y_n$ che contengano :

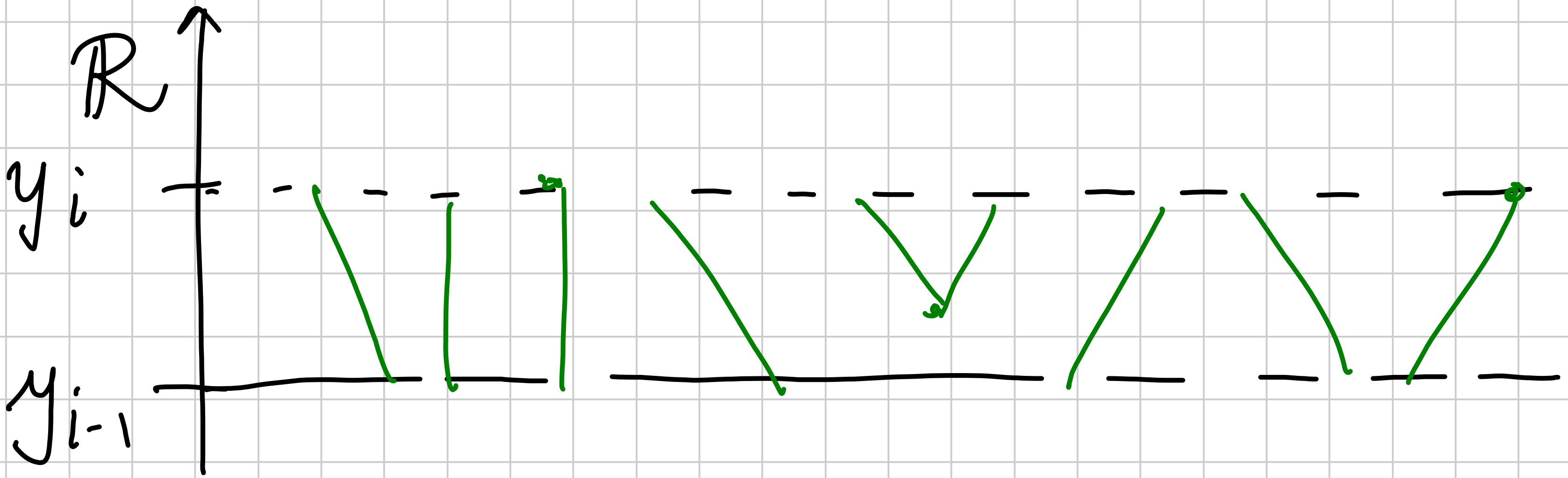


- tutte le altezze dei vertici che non sono min/max loc
- una altezza subito sopra e una subito sotto ogni max/min loc (no sotto per il minimo min, no sopra per il massimo max)





R
y_i
y_{i-1}



R
y_i
y_{i-1}

Chiamo γ_i $\quad \gamma \cap \{y \leq y_i\} \cup$ segmenti orizz ad altezza y_i in modo che diventi 1-var. PL

Esercizio:

$$\#(\gamma \cap \{y = y_i\}) \in 2\mathbb{Z}$$

y_i

$\gamma_i = \{$

Provo ricorsivamente che $\mathbb{R}^2 \setminus \gamma_i$ è unione di D_i e E_i

$\overline{D_i}$ che è unione di dischi bucati

$\overline{E_i}$ che è $\mathbb{R}^2 \setminus$ dischi aperti

con $\partial D_i = \partial E_i = \gamma_i$.

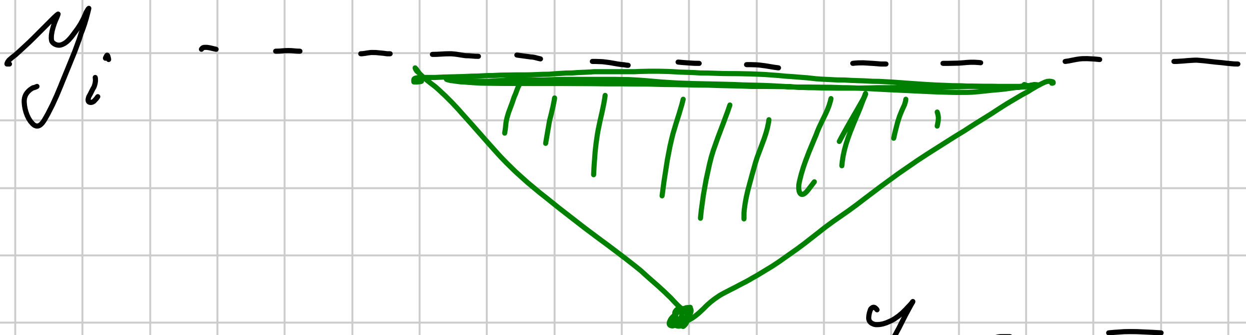
Ciò basta! per $i = N$

$\overline{D_N}$ unione dischi bucati, $\partial \overline{D_N} = \gamma_N = \gamma \cong S^1$

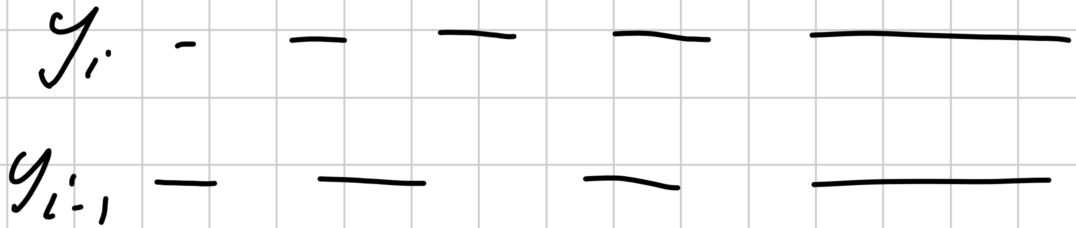
$\Longrightarrow \overline{D_N} \cong D^2$

Analogamente $\overline{E_N} \cong \mathbb{R}^2 \setminus \mathring{D}^2$.

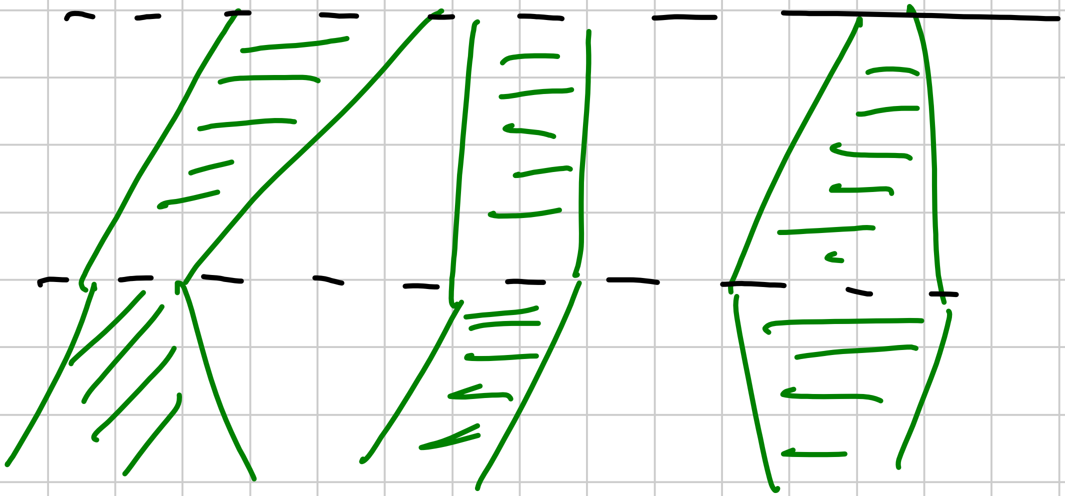
Passo base $i=1$



Passo induttivo:

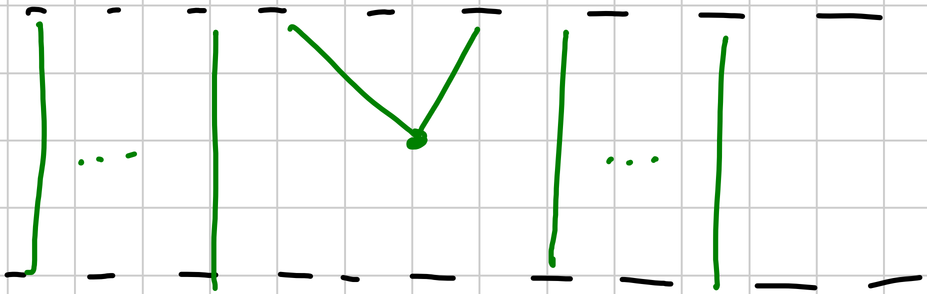


1)



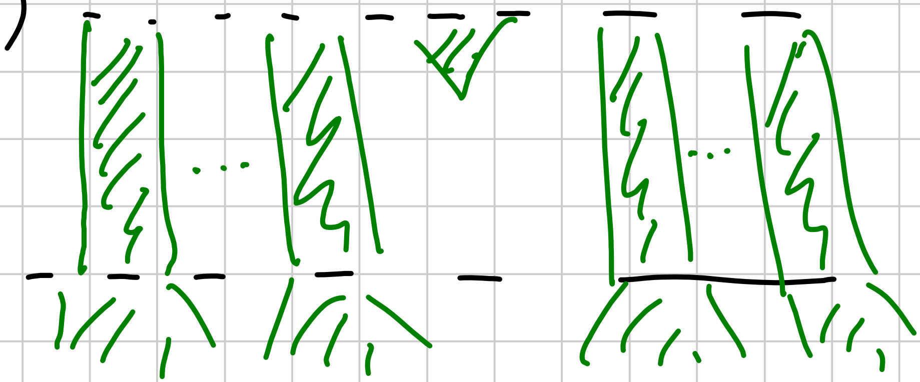
Tutto come prima

2)

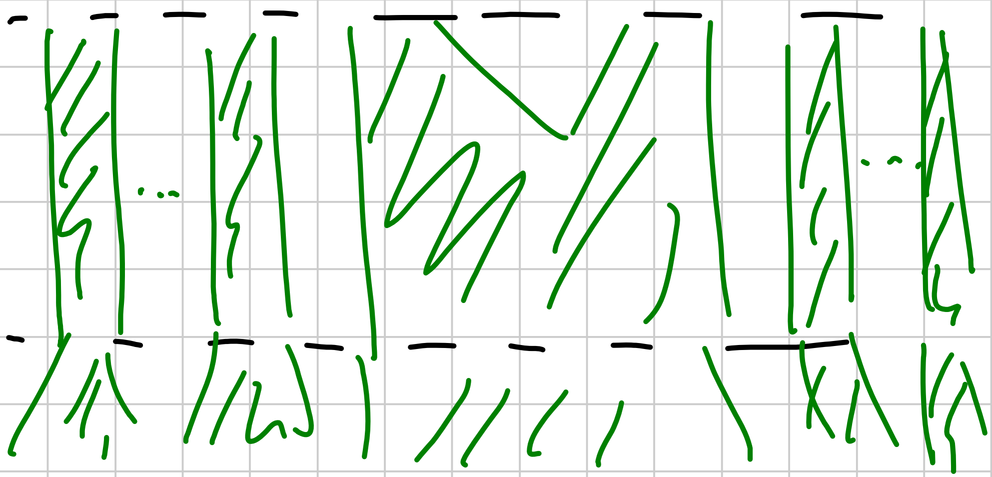


OK

2a)



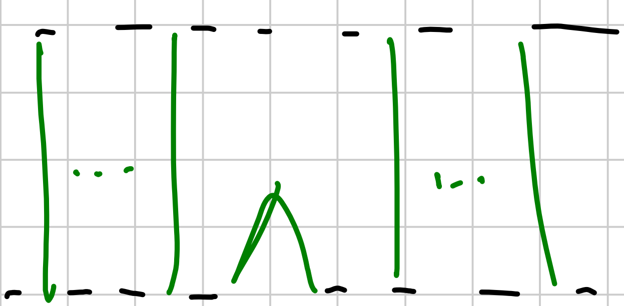
2b)



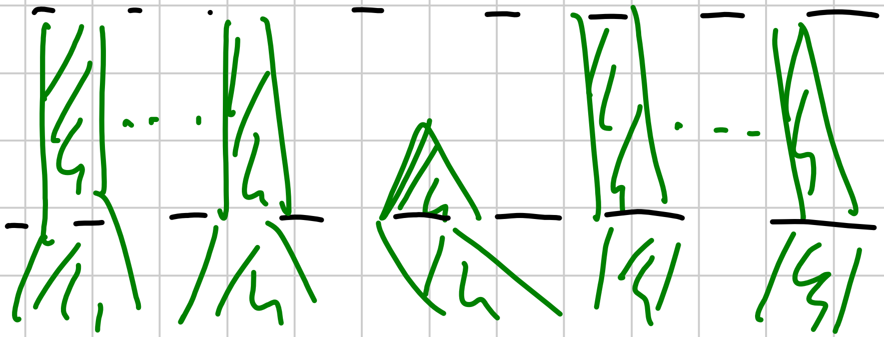
2a): tutto come prima + D^2

2b): tutto come prima

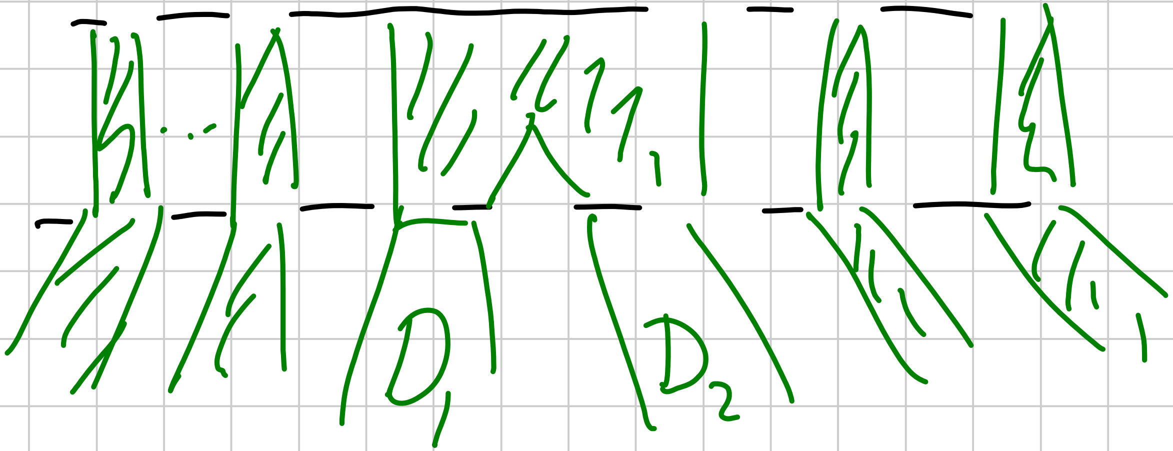
3)



3a)



3b)



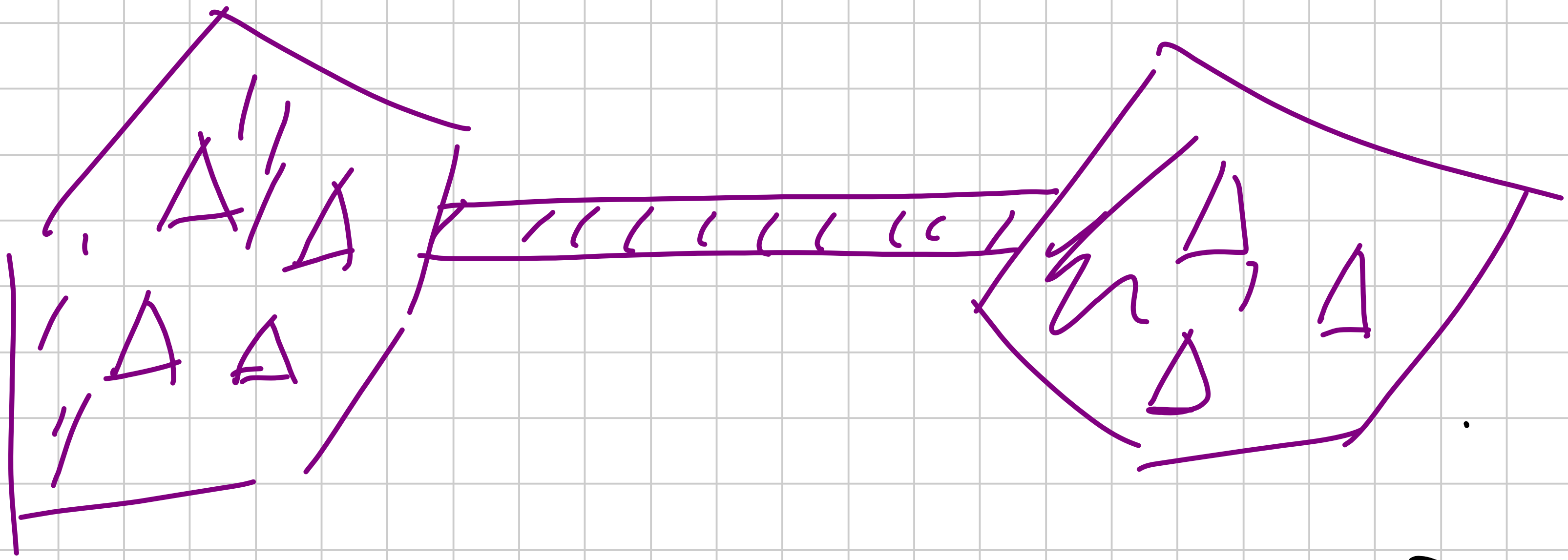
3a): tutto come prima

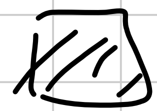
3b) *) $D_1 = D_2 =: D$

tutto come prima salvo nuovo bosco in D

*) $D_1 \neq D_2$: tutto come

prima salvo che D_1 e D_2 sono stati uniti:



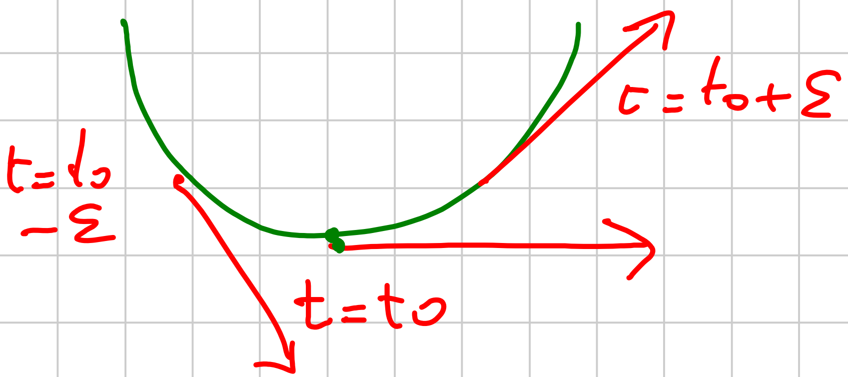


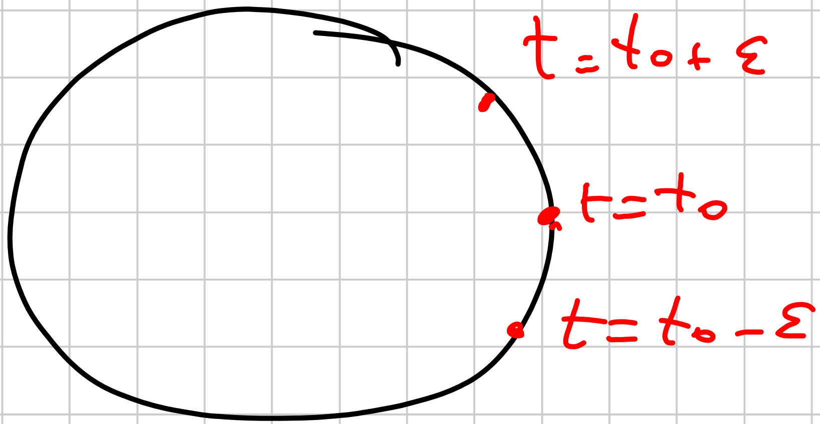
Variante Diff: considero $f: S^1 \to \mathbb{R}^2$ embedding.

Prendo $v \in S^1$ valore regolare per $\frac{f'}{\|f'\|}$ e per $-\frac{f'}{\|f'\|}$

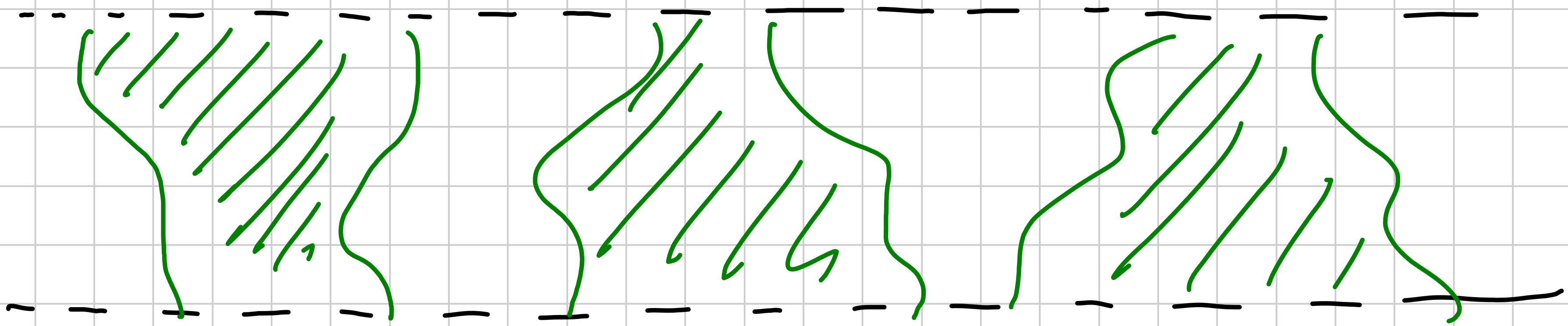
Wlog $v = (1,0)$; ora

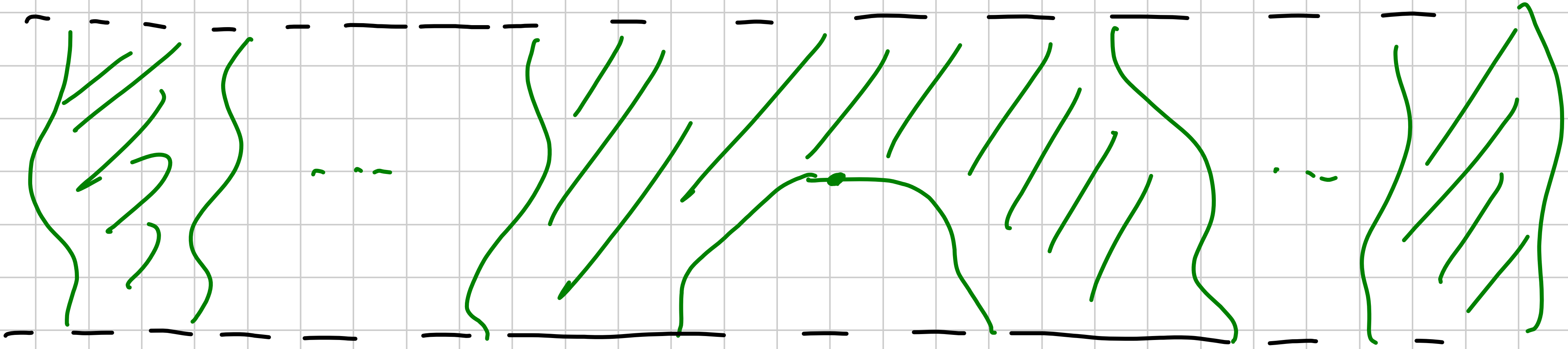
nei punti in cui f' è orizz ho max/min loc





Ora $y \circ f$ ha $\# < +\infty$ di max/min loc come soli punti critici; wlog max/min loc hanno y diverse: come prima suddivido





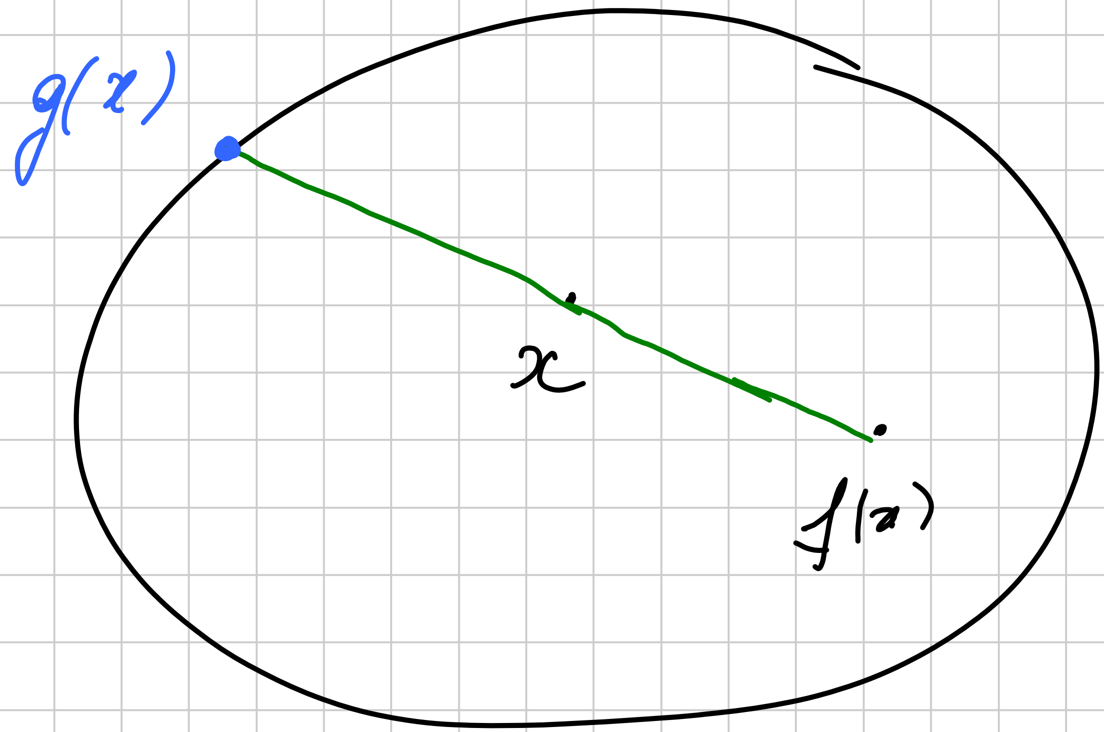
(Stessa strategia.) $\square$

Teo (punto fisso di Brouwer): $f: D^n \to D^n$ continua $\Longrightarrow \exists x \in D^n$ t.c. $f(x) = x$.

Dim: p.a.

$\exists g : D^n \to S^{n-1}$ t.c.

$g|_{S^{n-1}} = id_{S^{n-1}}$



$$
\begin{tikzcd}
S^{n-1}\times[0,1] \arrow[r, "G"] \arrow[d] & S^{n-1} \\
S^{n-1}\times[0,1]/S^{n-1}\times\{0\} \arrow[ur, bend right] \arrow[d, "{[y,t]\mapsto t\cdot y}"'] & \\
D^n \arrow[uur, bend right, "g"'] &
\end{tikzcd}
$$

$G(\cdot,0)$ costante $\quad G(y,1)=g(y)=y$

$\Rightarrow G$ è omotopia tra la costante e $id_{S^{n-1}}$

Assurdo $\deg(\text{cost}) = 0$

$\deg(id_{S^{n-1}}) = 1$.

$\blacksquare$

<u>Teo</u> (palla pelosa / "perchè servono le orecchie"):

$\nexists v : S^n \to \mathbb{R}^{n+1}$ mai nullo con $v(x) \perp x \ \forall x \in S^n$

se n è pari.

Lem: $f, g : X \to S^n$ mai antipodali tra loro

$\Longrightarrow f \simeq g$

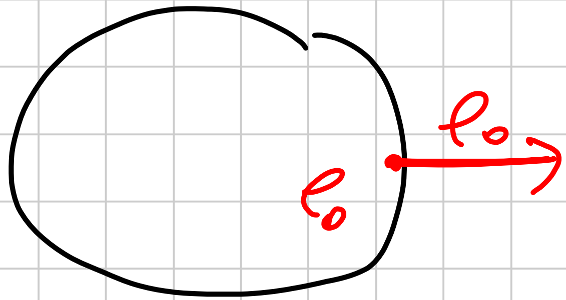
Pf: $$F(x,t) = \frac{(1-t)\, f(x) + t \cdot g(x)}{\|(1-t)\, f(x) + t \cdot g(x)\|} . \boxtimes$$

Lem: $\deg\left(-id_{S^n}\right) = (-1)^{n+1}$.

Dim: $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \|x\| = 1\}$

$e_0 \in S^n$ ed e_0 punta fuori a D^n in e_0

$e_1, \dots, e_n$ base di $T_{e_0} S^n = e_0^\perp$



$\Longrightarrow$ è base positiva poiché

ONF

$e_0, e_1, \dots, e_n$ è pos. in $\mathbb{R}^{n+1}$

$-e_0 = (-id_{S^n})(e_0)$ è esterno a D^n in $-e_0$;

$$d(-id_{S^n})(e_1, \dots, e_n) = (-e_1, \dots, -e_n)$$

$(-e_1, \dots, -e_n)$ è pos. per $T_{-e_0} S^n$

$\Longleftrightarrow$ ONF $\quad -e_0, -e_1, \dots, -e_n$ è pos per $\mathbb{R}^{n+1}$

Sì se $n+1$ è pari, no se $n+1$ dispari. $\boxed{//}$

<u>Conclusione</u>: se esistono $v : S^n \longrightarrow \mathbb{R}^{n+1}$ $\quad v(x) \neq 0 \; \forall x$

$v(x) \perp x \quad \forall x$

$$\frac{v(x)}{\|v(x)\|} \neq -x, x \quad \forall x$$

$$\Rightarrow \frac{v}{\|v\|} \simeq \mathrm{Id}_{S^n}, \quad \frac{v}{\|v\|} \simeq -\mathrm{Id}_{S^n}$$

assurdo perché hanno gradi $+1$ e -1. $\blacksquare$

— o —

Classificazione PL delle superfici.

Fatto (DURO): Hauptvermutung in dim $= 2$

Ogni 2-varietà di dimensione TOP ammette strutture PL e DIFF; due 2-varietà PL / DIFF omeomorfe sono PL-omeo / diffeomorfe.

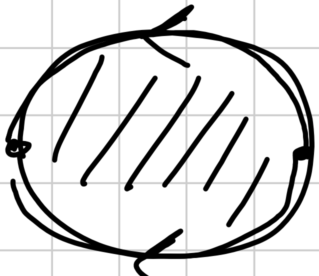
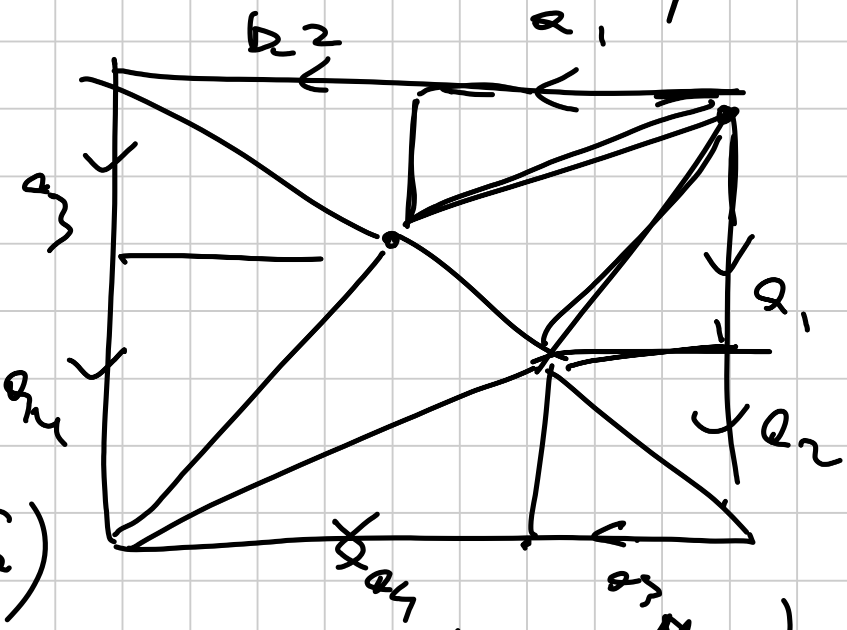
Cioè: per $n=2$ TOP = PL = DIFF

(Vero per $n=3$; $n=4$ TOP $\neq$ PL = DIFF

$n \geq 5$ tutti diversi).

———— o ————

Superficie: 2-var. chiuse connesse PL _ Esempi:

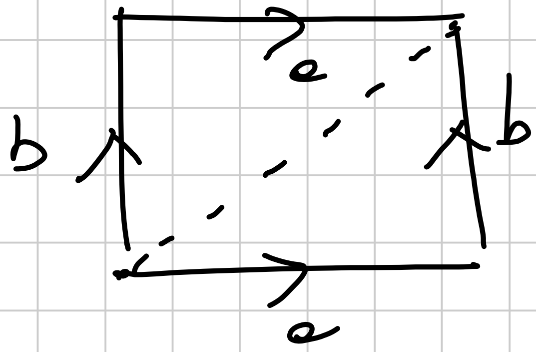
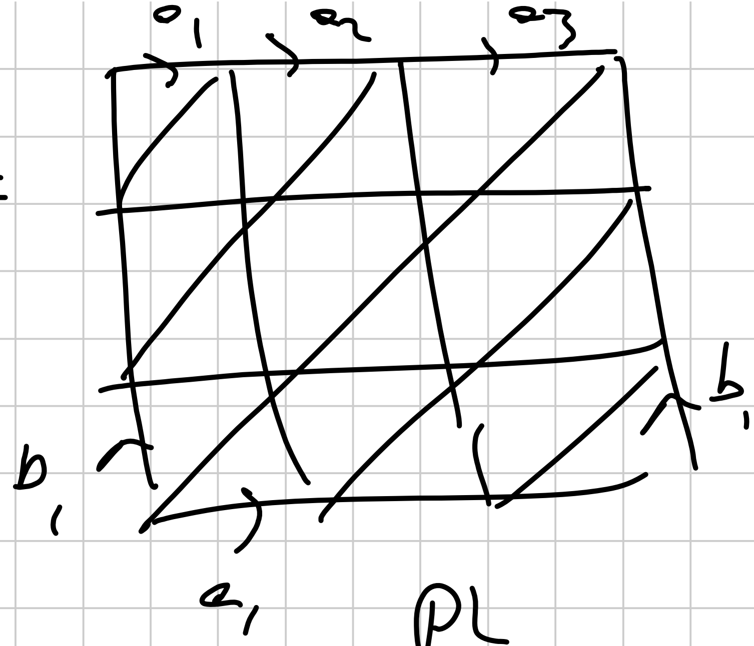
•) $S^2 = \partial \Delta_3 =$  $=$ 

DIFF PL TOP

"

$$(x, \sqrt{1-x^2}) \sim (x, -\sqrt{1-x^2})$$

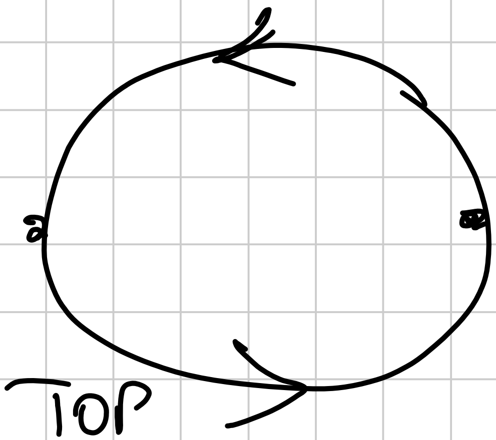
PL (astratto)
(realizzabile in $\mathbb{R}^N$)

•) $T = S^1 \times S^1$ =  = 

DIFF

TOP

PL

•) $\mathbb{P}^2 = S^2 / -\mathrm{id}$ =  = 

DIFF

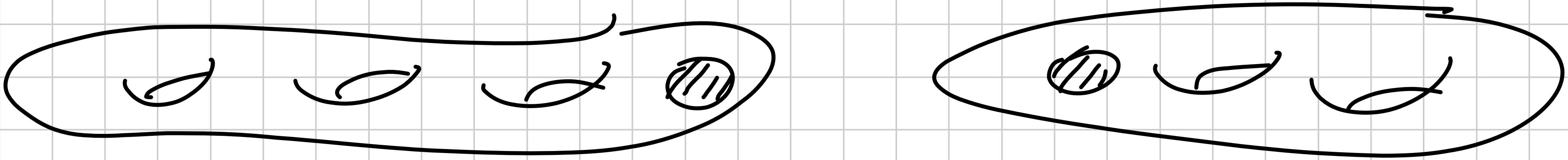
TOP

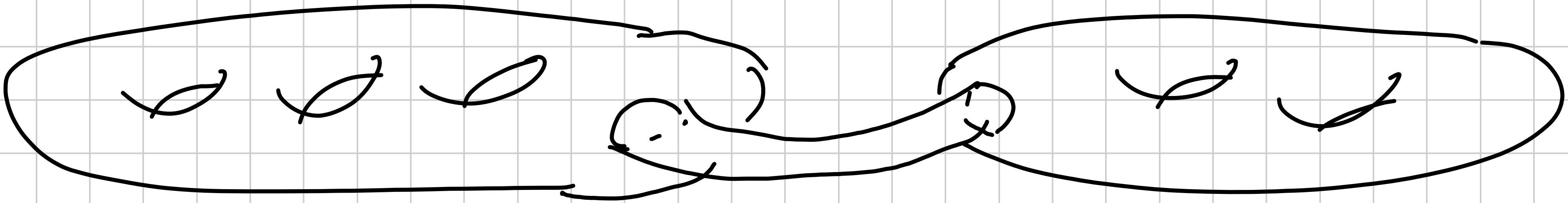
PL

Def: Somma connessa: $\Sigma_1 \# \Sigma_2 :=$ la superficie ottenuta come

$$\left(\Sigma_1 \setminus \mathring{T}_1\right) \cup_f \left(\Sigma_2 \setminus \mathring{T}_2\right)$$

$T_j \in \Sigma_j^{[2]}$ $\qquad$ $f : \partial T_1 \to \partial T_2$ omeo PL





<u>Fatto</u>: $\Sigma_1 \# \Sigma_2$ non dipende da T_1 e T_2; non dipende da f pre- o post-componendo con $\sigma \in \mathfrak{S}_3^+$; non dipende da f se per almeno una delle Σ_j esiste $\Delta : \Sigma_j \to \Sigma_j$ omeo PL con $\Delta(T_j) = T_j$ e Δ corrisponde

o permutazione dispari dei vertici di T_j –

<u>Teo</u>: ogni Σ è omeomorfa a una e una sola

$$S^2, \quad \underbrace{T \# \cdots \# T}_{n \geq 1}, \quad \underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{n \geq 1}$$

$n = 2k+1$ $\qquad$ $n = 2k+2$

$$\underbrace{T \# \cdots \# T}_{k} \# \mathbb{P}^2 \qquad k \geq 0 \qquad \underbrace{T \# \cdots \# T}_{k} \# K$$

(ed esse sono diverse tra loro) —