Alg. Lin. 17/12/13

29/11 5

$$\text{rank}\begin{pmatrix} 1 & 2 & 3 & 0 \\ \lambda & 0 & \lambda & \lambda+1 \\ 1 & \lambda & \lambda+1 & 0 \end{pmatrix}$$

$\lambda = -1$:
$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

rank ≥ 2

$\det = -2+3-1=0 \Rightarrow \text{rank} = 2$

$\lambda \neq -1$
$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ \lambda & 0 & \lambda & \lambda+1 \\ 1 & \lambda & \lambda+1 & 0 \end{pmatrix}$$

$\det = \lambda+1 = 0 \Rightarrow \text{rank} \geq 2$

$$\det\begin{pmatrix} 1 & 2 & 0 \\ \lambda & 0 & \lambda+1 \\ 1 & \lambda & 0 \end{pmatrix} = -(\lambda+1)(\lambda-2) \Rightarrow \text{rank} = 2 \text{ per } \lambda = 2$$

altrimenti rank $= 3$

$$\det\begin{pmatrix} 1 & 3 & 0 \\ \lambda & \lambda & \lambda+1 \\ 1 & \lambda+1 & 0 \end{pmatrix} = -(\lambda+1)(\lambda-2)$$

6/12 (6) eq. par $\longleftrightarrow$ eq. cart.

(d)

$$\begin{cases} (1+i)z_1 + 3i\,z_2 + (3-2i)z_3 + 5z_4 = 2i \\ (2+i)z_1 + 2i\,z_2 + (1-2i)z_3 + 3i\,z_4 = 1 \end{cases}$$

$\dim = 4-2 = 2$

$\neq \emptyset$

$$\frac{1}{5}\begin{pmatrix} -1-4i \\ 0 \\ -3+3i \\ 0 \end{pmatrix} + \operatorname{Span}\left(\begin{pmatrix} 2-3i \\ 5 \\ -5-4i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1-18i \\ 9+10i \\ 2-3i \end{pmatrix}\right)$$

$3i(1-2i)-2i(3-2i) = 3i+6-6i-4$

$-\big((1+i)(1-2i)-(2+i)\cdot(3-2i)\big) = -(1-2i+i+2-6+4i-3i-2) = 5$

(e) $\begin{pmatrix} -1 \\ 1+i \\ 2i \end{pmatrix} + \mathrm{Span}\begin{pmatrix} 1+i \\ -4i \\ 7 \end{pmatrix}$ $\dim = 1$

$$\begin{cases} 4iz_1 + (1+i)z_2 = -2i \\ 7z_2 + 4i\, z_3 = 7i-1 \end{cases}$$

$-4i + (1+i)^2 = -2i$

$7+7i-8 = 7i-1$

(f) $\begin{pmatrix} 1 \\ 3i \\ -4i \\ -4 \end{pmatrix} + \operatorname{Span}\begin{pmatrix} 6+i \\ 1+i \\ 2-i \\ 5+i \end{pmatrix}$

$$\begin{cases} (1+i)z_1 - (6+i)z_2 = 4-17i \\ (2-i)z_1 - (6+i)z_3 = \dots \\ (5+i)z_1 - (6+i)z_4 = \dots \end{cases}$$

$$1\cdot(1+i) - (6+i)\cdot 3i = 1+i-18i+3 =$$

(g) $\begin{pmatrix} i \\ 3 \\ 1 \end{pmatrix} + \operatorname{Span}\left(\begin{pmatrix} 2 \\ 1-i \\ 3i \end{pmatrix}, \begin{pmatrix} -5i \\ 2 \\ 1+2i \end{pmatrix}\right)$ $\dim = 2$

$\Rightarrow$ 1 equaz.

$$(1-i)(1+2i) - 3i\cdot 2 = 1-i+2i+2-6i = 3-5i$$

$$-\left(2(1+2i) - 3i(-5i)\right) = -(2+4i-15) = 13-4i$$

$$2\cdot 2-(1-i)(-5i) = 4+5i+5 = 9+5i$$

$$(3-5i)z_1+(13-4i)z_2+(9+5i)z_3 = 53-4i$$

$3i+5$ $\quad$ $39-12i$ $\quad$ $9+5i$

(h) $\begin{pmatrix}-1\\ i\\ 1+i\\ 1\end{pmatrix} + \mathrm{Span}\left(\begin{pmatrix}i\\ 3-i\\ 4-i\\ 2+i\end{pmatrix}, \begin{pmatrix}1+i\\ 2\\ 3i\\ i-3\end{pmatrix}\right)$ $\quad$ dim $=2$

2 equaz.

$$\begin{cases}\begin{pmatrix}(3-i)\cdot 3i\\ -(4-i)\cdot 2\end{pmatrix} z_1 - (i\cdot 3i-(4-i)(1+i))z_2 + (i\cdot 2-(3-i)(1+i))z_3 + 0\cdot z_4 = \dots \\ 0\cdot z_1 + \begin{pmatrix}(4-i)(i-3)\\ -(2+i)3i\end{pmatrix} z_2 - \begin{pmatrix}(3-i)(i-3)\\ -(2+i)\cdot 2\end{pmatrix} z_3 + \begin{pmatrix}(3-i)3i\\ -(4-i)\cdot 2\end{pmatrix} z_4 = \dots\end{cases}$$

13/12 1 (a) $z^2 + iz + 2 = 0$

$$\left(\begin{array}{c} az^2 + bz + c = 0 \\ z_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} \\ \Delta = b^2 - 4ac \end{array} \right)$$

$\Delta = -1 - 8 = -9$

$$z_{1,2} = \frac{-i \pm 3i}{2} = \begin{cases} i \\ -2i \end{cases}$$

$z^2 + bz + c = 0$

$b = -(z_1 + z_2) \qquad c = z_1 \cdot z_2$

(b) $z^2 - 3z + 3 + i = 0$

$\Delta = 9 - 12 - 4i = -3 - 4i$

Cerco le radici: $\pm(a+ib)$; voglio che $(a+ib)^2 = -3-4i$, cioè $a^2 - b^2 + 2iab = -3 - 4i$, cioè

$$\begin{cases} a^2 - b^2 = -3 \\ ab = -2 \end{cases} \qquad a = 1 \quad b = -2$$

$$z_{1,2} = \frac{3 \pm (1-2i)}{2} \quad \begin{matrix} \nearrow 2 - i \\ \searrow 1 + i \end{matrix}$$

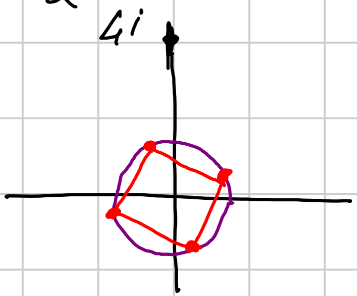
(c) $z^2+(1-7i)z-22+7i=0$

$\Delta = 1-14i-49+88-28i = 40-42i$

$\begin{cases} a^2-b^2=40 \\ ab=-21 \end{cases}$ $\quad a=7 \quad b=-3$

$z_{1,2} = \dfrac{-1+7i \pm (7-3i)}{2} = \begin{cases} 4-5i \\ -3-2i \end{cases}$

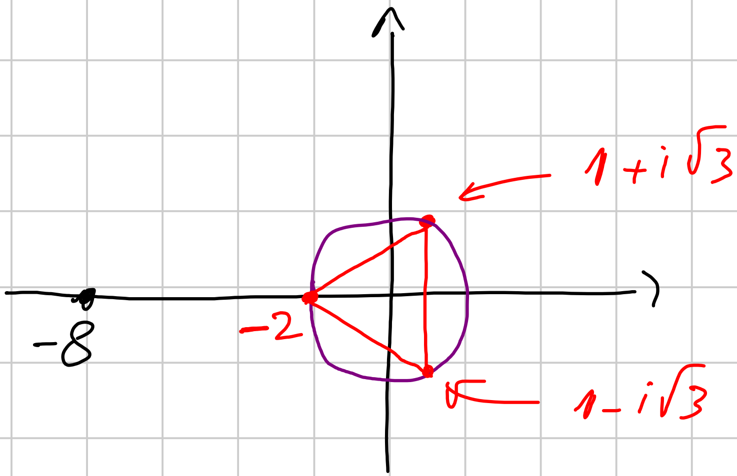
(e) $z^4=4i$



$\sqrt{2}\cdot e^{i\left(\frac{\pi}{8}+k\frac{\pi}{2}\right)}$ $\quad k=0,1,2,3$

(d) $z^3 = -8$

$$2 \cdot e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)} \quad k = 0,1,2$$



(f) $3(1+i)z^3 - 4(3+i)z^2 + (13-i)z + 2(1-2) = 0$

$z=1$: $\underline{3} + \underline{3i} \underline{-12} \underline{-4i} \underline{+13} \underline{-1} \underline{+2i} - \underline{4} = 0$

$3(1+i)z^3 + \dots$

$= (z-1)\left(3(1+i)z^2 + a\cdot z \quad -2(i-2)\right)$

$-12-4i = -3-3i+a$

$\Longrightarrow a = -9-i$

$\Delta = 81+18i-1+24(1+i)(i-2) = \dots = 8-6i$

$\begin{cases} \alpha^2-\beta^2 = 8 \\ \alpha\beta = -3 \end{cases} \quad \alpha=3,\ \beta=-1$

$z_1 = 1, \quad z_{2,3} = \dfrac{9+i \pm (3-i)}{6(1+i)} \begin{cases} 1-i \\ \frac{1}{3}(2-i) \end{cases}$

(g) $z^3\bar{z} + 8i = 2z(z+2i\bar{z})$

Se sostituisco $z = x+iy$ $(\bar{z} = x-iy)$ viene un sistema di 2 equazioni polinomiali di grado 4 in x, y. Invece:

$$|z|^2 \cdot z^2 + 8i - 4z^2 - 4i|z|^2 = 0$$

$$z^2(|z|^2-4) - 4i(|z|^2-4) = 0$$

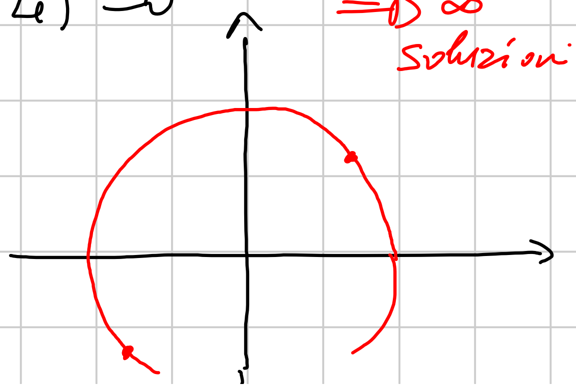
$$(z^2-4i)(|z|^2-4) = 0$$

$$z^2 = 4i \quad \text{o} \quad |z|^2 = 4$$

$$\pm\sqrt{2}(1+i) \quad \text{o} \quad |z| = 2$$

compresi in $|z|=2$

$\Rightarrow \infty$ soluzioni



(L'equazione originale non era polinomiale.)

TUTTE LE REGOLE PER GLI ESAMI SONO SCRITTE SUL MIO SITO

(DIFFIDARE DELLE VOCI)

(2) (a) $z^3 + (2-4i)z^2 - (4+8i)z - 8$ $\qquad z_1 = -2$

$$
\begin{array}{c|ccc|c}
 & 1 & 2-4i & -4-8i & -8 \\
-2 & & -2 & 8i & 8 \\
\hline
 & 1 & -4i & -4 & /
\end{array}
$$

$p(z) = (z+2)(z^2 - 4iz - 4) = (z+2)(z-2i)^2$

$z_1 = -2$ $\quad$ molt$(z_1) = 1$; $\quad z_2 = 2i$ $\quad$ molt$(z_2) = 2$.

(b) $4z^3+(8i-4)z^2-(5+4i)z+(1-i)$ $\qquad z_1=-i/2$

$(2z-i)(2z^2+\ \ldots\qquad +1+i)$

(c) $z^4+(1-i)z^3+(1+3i)z^2+(8-i)z-5i$ $\qquad z_1=i$

$=(z-i)(z^3+\quad z^2+(1+4i)\ z\ +5)$

$-i\qquad -1\quad +i-4\qquad +5\qquad (z=i)$

$=(z-i)^2(z^2+(1+i)z+5i)$

$z_1=i$ molt. $=2$

$z_2 = -2+i \qquad \text{molt} = 1$

$z_3 = 1-2i \qquad \text{molt} = 1$

(d) $p(z) = \left(z-(1-i)\right)^3 \cdot (z+2-i)$

$z_1 = 1-i \qquad \text{molt} = 3$

$z_2 = -2+i \qquad \text{molt} = 1$

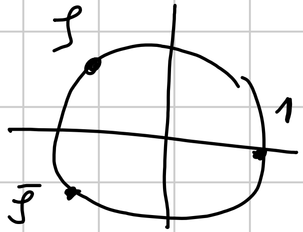
③ $z = \frac{1}{25}(1-3i)^3 + \frac{1+i}{(1-2i)^2} + i^{83} \qquad Re(z), Im(z) ?$

$$z = \frac{1}{25}\left(1 - 9i - 27 + 27i\right) + (1+i)\left(\frac{1+2i}{5}\right)^2 - i$$

$$= \frac{1}{25}\left(-26 + 18i + \underbrace{(1+i)(-3+4i)}_{-7+i} - 25i\right) = \underbrace{-\frac{33}{25}}_{\text{Re}} \underbrace{- \frac{6}{25}}_{\text{Im}} i$$

4. $\xi = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i \cdot 2\pi/3}$ $\qquad \xi^{91} \overset{?}{=} \xi \; ; \; \xi^{90} \overset{?}{=} 1$

$$\xi^{90} = (\underbrace{\xi^3}_{=1})^{30} \quad \text{Sì}$$



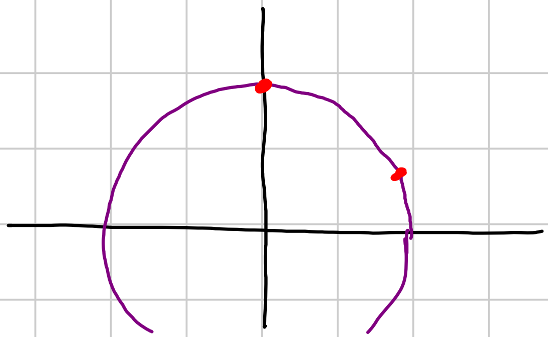
⑤ $z^7+z^6+4z+4=0$

$(z^6+4)(z+1)=0$

$z_0,\dots,z_5$ radici seste di -2

$z_6=-1$

$$z_{0,\dots,5} = \sqrt[6]{2}\cdot e^{i\left(\pi/6+k\cdot\frac{\pi}{3}\right)} \qquad k=0,\dots,5$$



Soluz. con parte reale nulla $\pm \sqrt[6]{2} \cdot i$

$\boxed{1}$ ① 8 gen di $W = \{ z \in \mathbb{C}^6 : i z_3 = (1-i) z_1 + 7 z_5 \}$

quanti ne scarto per base?

$$8 - \dim = 8 - (6 - 1) = 3$$

② $f : \{ x \in \mathbb{R}^4 : 3x_1 \ldots = 0 \} \to \mathbb{R}^7$ lin. iniett.

$Z \oplus \operatorname{Im} f = \mathbb{R}^7$; $\dim Z$?

$$\underset{\text{Grassmann}}{\dim Z = 7 - \dim \operatorname{Im} f} \underset{\substack{f \text{ iniett} \\ + \text{ formula} \\ \dim}}{=} 7 - \underbrace{\dim(\operatorname{Dom}(f))}_{4-1} = 4$$

⑥ Trovare base di $V = \{ z \in \mathbb{C}^3 : (1+i)z_1 + (2-i)z_2 + (1+3i)z_3 = 0 \}$ in cui ogni vett. ha una compon. 0 e una 1

$$\underset{v_1}{\begin{pmatrix} 0 \\ 1 \\ -\frac{2-i}{1+3i} \end{pmatrix}} \quad \underset{v_2}{\begin{pmatrix} 1 \\ 0 \\ -\frac{1+i}{1+3i} \end{pmatrix}} \quad \underset{v_3}{\begin{pmatrix} 0 \\ -\frac{1+3i}{2-i} \\ 1 \end{pmatrix}} \quad \underset{v_4}{\begin{pmatrix} 1 \\ \cdots \\ 0 \end{pmatrix}} \quad \underset{v_5}{\begin{pmatrix} \cdots \\ 0 \\ 1 \end{pmatrix}} \quad \underset{v_6}{\begin{pmatrix} \cdots \\ 1 \\ 0 \end{pmatrix}}$$

Va bene ogni scelta di (v_i, v_j) tranne
(v_1, v_3) (v_2, v_5) (v_4, v_6) –

$\boxed{12}$ (4) $f: \mathbb{R}^4 \to \mathbb{R}^7$ lin. $f(3e_1 - 5e_2) = f(4e_3 + 7e_4)$
$W \oplus \operatorname{Im} f = \mathbb{R}^7$ $\quad \dim W = ?$

$$\dim W = 7 - \dim \operatorname{Im} f = 7 - \underbrace{(4 - \underbrace{\dim \operatorname{Ker} f}_{1 \le r \le 4})}_{0 \le r \le 3}$$

$\iff f(3e_1 - 5e_2 - 4e_3 - 7e_4) = 0$

$\Rightarrow 4 \le \dots \le 7$

$\boxed{3}$ ③ $f: \mathbb{C}^7 \to \{z \in \mathbb{C}^4 : (1-i)z_1 + 2z_3 - z_4 = 0\}$

lin. surj.

$W \subset \mathbb{C}^7$, $W \cap \operatorname{Ker} f = \{0\}$; $\dim W = ?$

$$0 \le \dim W \le \underbrace{7 - \underbrace{\dim(\operatorname{Ker} f)}_{7 - \dim \operatorname{Im} f}}_{\dim \operatorname{Im} f = 4 - 1 = 3}$$

⑤ Risolvere
$$\begin{cases} 3x - 2y + 5z = 10 \\ -4x + 5y + 2z = -7 \\ 2x + y + 12z = 13 \end{cases}$$

$$\det\begin{pmatrix} 3 & -2 & 5 \\ -4 & 5 & 2 \\ 2 & 1 & 12 \end{pmatrix} = \cdots = 0$$

($\Rightarrow$ nessuna o infinite soluz) (rank $= 2$)

Parte omogenea: $\mathrm{III} = 2\cdot \mathrm{I} + \mathrm{II}$

Stessa relaz. vale per termini noti?

- Sì: ∞ soluz (dim $= 1$)
- ~~No: nessuna soluz.~~

$$\begin{cases} 3x - 2y + 5z = 10 \\ -4x + 5y + 2z = -7 \\ \end{cases}$$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mathrm{Span}\begin{pmatrix} -29 \\ -26 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 25 \\ -6 \end{pmatrix} + \mathrm{Span}\begin{pmatrix} 29\sqrt{3}\cdot\pi \\ 26\cdot\sqrt{3}\cdot\pi \\ -7\cdot\sqrt{3}\cdot\pi \end{pmatrix}$$

Domani: $\boxed{4}$ 4 $\boxed{5}$ 3, 6 $\boxed{6}$ 7

$\boxed{7}$ 1 2 3 5 $\boxed{9}$ 3 4 6 $\boxed{12}$ 3

$\boxed{13}$ 6 $\boxed{15}$ 1, 3, 6 $\boxed{16}$ 2, 4 $\boxed{17}$ 5 $\boxed{22}$ 7