Esercizi 22/11/13

4. (c) Supponiamo che il sistema fosse:

$$\begin{cases} 7x - y + 2z = 4 \\ 2x + 3y - 5z = -3 \\ x - 10y + 17z = 13 \end{cases}$$

← ho cambiato il termine noto

$$A = \begin{pmatrix} 7 & -1 & 2 \\ 2 & 3 & -5 \\ 1 & -10 & 17 \end{pmatrix} \quad \det A = 0.$$

$$\begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -10 \end{pmatrix} \text{ indipendenti} \Rightarrow \dim \operatorname{Im}(f_A) = 2.$$

$$\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = a \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} -1 \\ 3 \\ -10 \end{pmatrix} \iff$$

$$\begin{cases} 7a - b = 4 \\ 2a + 3b = -3 \\ a - 10b = 13 \end{cases}$$

$\left.\begin{matrix} 7a - b = 4 \\ 2a + 3b = -3 \end{matrix}\right\} \longrightarrow a = 9/23, \quad b = -29/23$

$9/23 - 10 \cdot \left(-\frac{29}{23}\right) = 13$ ✓

$$\Rightarrow \begin{pmatrix} 4 \\ -3 \\ 13 \end{pmatrix} \in \operatorname{Im}(f_A) \Rightarrow \text{il sistema ha soluzioni.}$$

$$\begin{pmatrix} 4 \\ -3 \\ 13 \end{pmatrix} = A \cdot \begin{pmatrix} 9/23 \\ -29/23 \\ 0 \end{pmatrix}$$

x_0 soluzione particolare del sistema

$$\left\{ x : Ax = \begin{pmatrix} 4 \\ -3 \\ 13 \end{pmatrix} \right\} = \left\{ x_0 + x : Ax = 0 \right\}$$

Risolviamo il sistema omogeneo associato: ②

Poiché $\dim \operatorname{Im} f_A = 2$, $\dim \operatorname{Ker} f_A = 1$.

Quindi $\operatorname{Ker} f_A$ è generato da una soluzione $\neq 0$ del sistema omogeneo.

$$\begin{pmatrix} 2 \\ -5 \\ 17 \end{pmatrix} \stackrel{?}{=} a \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 3 \\ -10 \end{pmatrix} \iff \begin{cases} 7a - b = 2 \\ 2a + 3b = -5 \\ a - 10b = 17 \end{cases}$$

$\xrightarrow{\text{sol.}}$ $a = 1/23$, $b = -39/23$ $\quad \begin{pmatrix} 1/23 \\ -39/23 \\ -1 \end{pmatrix}$ sol. del sistema omogeneo.

$$\Rightarrow \{\text{soluzioni}\} = \left\{ \begin{pmatrix} 9/23 \\ -29/23 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ -39t \\ -23t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Esercizio 6

$$A = \begin{pmatrix} 1-t & 2 & -2 \\ 1+t & 3 & 1 \\ 7 & 12 & 2t \end{pmatrix} \leadsto \det A = -10t^2 - 10t + 20$$

$$\det A = 0 \iff t^2 + t - 2 = 0$$

$$\iff t = \frac{-1 \pm \sqrt{1+8}}{2} \begin{array}{l} \nearrow -2 \\ \searrow 1 \end{array}$$

$\Rightarrow$ per $t \neq 1, -2$ il sistema ha una sola soluzione perché A è invertibile.

Se $t = -2$:

$$A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 3 & 1 \\ 7 & 12 & -4 \end{pmatrix}. \quad \text{Sappiamo } \dim \operatorname{Im} f_A \leq 2$$

$$\begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 12 \end{pmatrix} \text{ indipendenti} \Rightarrow \dim \operatorname{Im} f_A = 2$$

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \overset{?}{=} a \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + b \cdot \begin{pmatrix} 2 \\ 3 \\ 12 \end{pmatrix}$$

$$\begin{cases} 3a + 2b = 1 \\ -a + 3b = -2 \\ 7a + 12b = -1 \end{cases} \qquad I + 3 \cdot II : \quad 11b = -5 \qquad a = -\tfrac{15}{11} + \tfrac{22}{11} = \tfrac{7}{11}$$

$$\frac{49 - 60}{11} = -1 \quad \checkmark$$

$$\Rightarrow \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \in \operatorname{Im} f_A$$ e il sistema ammette infinite soluzioni.

<u>$t = 1$</u> :

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 3 & 1 \\ 7 & 12 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in \operatorname{Im} f_A \,?$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 2b \\ 2a + 3b \\ 7a + 12b \end{pmatrix} \begin{array}{l} \rightarrow \; b = 1/2 \\ \rightarrow \; a = -1/4 \end{array}$$

$$-\frac{7}{4} + \frac{12}{2} = \frac{24}{4} = \frac{17}{4} \neq -1$$

$\Rightarrow$ il sistema non ammette soluzioni.

Esercizio 7 :

$$\det\begin{pmatrix} -3 & -1 & 2 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 3 & 2 & -1 \\ -4 & -2 & 1 & 3 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 & 2 & -1 \\ -3 & -1 & 2 & 1 \\ 2 & 1 & -1 & 1 \\ -4 & -2 & 1 & 3 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 3 & 2 & -1 \\ 0 & 8 & 8 & -2 \\ 0 & -5 & -5 & 3 \\ 0 & 10 & 9 & -1 \end{pmatrix} \begin{matrix} \\ \leftarrow II + 3\cdot I \\ \leftarrow III - 2\cdot I \\ \leftarrow IV + 4\cdot I \end{matrix}$$

$$= 2\cdot\det\begin{pmatrix} 4 & 4 & -1 \\ -5 & -5 & 3 \\ 10 & 9 & -1 \end{pmatrix} =$$

$$= 2\cdot\big(4\cdot(-5)(-1) + 4\cdot 3\cdot 10 + (-5)\cdot 9\cdot(-1)$$
$$- 10(-5)(-1) - 4\cdot 9\cdot 3 - (-5)(4)(-1)\big)$$

$$= 14$$

Esercizi 29/11/2013

1. (a) $(4312) \to (1342) \to (1243) \to (1234)$

$\Rightarrow (4312)$ è prodotto di 3 scambi $\to$

$\sigma(4312) = (-1)^3 = -1$.

(b) $(41523) \to (14523) \to (12543) \to (12345)$

$\sigma = (-1)^3 = -1$

(c) $(361524) \to (163524) \to (123564) \to (123465)$

$\to (123456) \quad \Rightarrow \sigma = (-1)^4 = 1$.

(d) $(5417236) \to (1457236) \to (1257436) \to$

$(1237456) \to (1234756) \to (1234576) \to (1234567)$

$\Rightarrow \sigma = (-1)^6 = 1$.

2. (a) $$\frac{\det B}{\det A} = \frac{\det(3v_1 - 2v_2, -5v_1 + 4v_2)}{\det(v_1, v_2)} =$$

$$= \frac{-15\overset{0}{\det(v_1, v_1)} + 12\det(v_1, v_2) + 10\det(v_2, v_1) - 8\overset{0}{\det(v_2, v_2)}}{\det(v_1, v_2)}$$

$\det(v_2, v_1) = -\det(v_1, v_2)$

$= 2$

(b) $\det B = 6\det(v_2, v_1, v_3) - \det(v_2, v_3, v_1)$

$-4\det(v_3, v_1, v_2) =$

$= -6\det(v_1, v_2, v_3) - \det(v_1, v_2, v_3) - 4\det(v_1, v_2, v_3)$

$\Rightarrow \frac{\det B}{\det A} = -11$

(c) $\det B = \det(v_3, v_1, 2v_2, -2v_4) + \det(-v_1, -3v_4, 2v_2, v_3) =$

$= -4\det(v_3, v_1, v_2, v_4) + 6\det(v_1, v_4, v_2, v_3)$

$= (-4+6)\det(v_1, v_2, v_3, v_4)$

$\Rightarrow \dfrac{\det B}{\det A} = 2.$

3. (a) $\begin{pmatrix} 3 & -2 \\ 7 & 1 \end{pmatrix}^{-1} = \dfrac{1}{3+14} \cdot \begin{pmatrix} 1 & 2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 1/17 & 2/17 \\ -7/17 & 3/17 \end{pmatrix}$

(b) $\begin{pmatrix} 1+t & t-9 \\ 1-t & t \end{pmatrix}^{-1} = \dfrac{1}{\cancel{t}+t^2-\cancel{t}+t^2+9-9t} \cdot \begin{pmatrix} t & 9-t \\ t-1 & 1+t \end{pmatrix}$

$= \dfrac{1}{2t^2-9t+9} \cdot \begin{pmatrix} t & 9-t \\ t-1 & 1+t \end{pmatrix}$

(c) $\det \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -1 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 & 0 \\ 1 & -2 & 1 \\ 4 & 1 & 0 \end{pmatrix} = +1$

${}^t\begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 13 & 9 \\ 0 & 1 & 1 \\ 1 & -10 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 13 & 1 & -10 \\ 9 & 1 & -7 \end{pmatrix}$

(d) $\det \begin{pmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \overset{4^a c \to 4^a c + 2\cdot 1^a c}{=} \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 5 \\ 3 & 0 & 1 & 5 \\ 0 & 2 & 1 & 1 \end{pmatrix} =$

$= \det \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix} \underset{3^a r \to 3^a r - 2\cdot 1^a r}{=} \det \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 1 & -9 \end{pmatrix} = -14$

Calcolando tutti i complementi algebrici si ottiene:

$$^t\begin{pmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix}^{-1} = -\frac{1}{14}\begin{pmatrix} 0 & -7 & 7 & 7 \\ -4 & -4 & 10 & -2 \\ -2 & 5 & -9 & -1 \\ 2 & -5 & -5 & 1 \end{pmatrix}$$

E infine
$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 2/7 & 1/7 & -1/7 \\ 1/2 & 2/7 & -5/14 & 5/14 \\ -1/2 & -5/7 & 9/14 & 5/14 \\ -1/2 & 1/7 & 1/14 & -1/14 \end{pmatrix}$$

4. (a) $\begin{pmatrix} 2 & 1 & 5 & 5 \\ 1 & 2 & 1 & 7 \\ 3 & 1 & 8 & 6 \end{pmatrix}$ (non-singolare: minore $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$)

$$\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & 2 & 1 \\ 3 & 1 & 8 \end{pmatrix} = \det\begin{pmatrix} 0 & -3 & 3 \\ 1 & 2 & 1 \\ 0 & -5 & 5 \end{pmatrix}$$

$1^a r \to 1^a r - 2 \cdot 2^a r$
$3^a r \to 3^a r - 3 \cdot 2^a r$

$$= \det\begin{pmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 0 & -5 & 0 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & 2 & 7 \\ 3 & 1 & 6 \end{pmatrix} \overset{\text{IR}-2\cdot\text{IIR},\ \text{IIIR}-3\cdot\text{IIR}}{=} \det\begin{pmatrix} 0 & -3 & -9 \\ 1 & 2 & 7 \\ 0 & -5 & -15 \end{pmatrix} = \det\begin{pmatrix} 0 & -3 & 0 \\ 1 & 2 & 13 \\ 0 & -5 & 0 \end{pmatrix} = 0$$

$\Rightarrow$ per il teo. degli orlati $rg = 2$.

(b) $\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 0 & 1 & 0 \\ 3 & -6 & 1 & 4 & 3 \end{pmatrix}$

$$\det\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 3 \end{pmatrix} \overset{\text{IR}-2\text{IIR},\ \text{IIIR}-4\cdot\text{IIR}}{=} \det\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} = 1 \neq 0$$

$\Rightarrow rg = 3$