Esercizi 15/11/13

1 (b) [non svolto in aula] $\quad M = [id_{\mathbb{R}^3}]^{\mathcal{B}}_{\mathcal{B}'}$

Dobbiamo determinare le coordinate dei vettori di $\mathcal{B}'$ nella base $\mathcal{B}$:

$$\begin{cases} a = x_1 \\ 2a - b + c = x_2 \\ b + 2c = x_3 \end{cases} \Leftrightarrow \begin{cases} a = x_1 \\ -b + c = x_2 - 2x_1 \\ b + 2c = x_3 \end{cases}$$

$$\begin{cases} a = x_1 \\ c = 1/3\,(-2x_1 + x_2 + x_3) \\ b = c - x_2 + 2x_1 = \frac{1}{3}(4x_1 - 2x_2 + x_3) \end{cases}$$

$$\Rightarrow \quad M = \begin{pmatrix} -1 & 2 & 1 \\ -1 & 10/3 & 2 \\ 1 & -5/3 & 0 \end{pmatrix}$$

$$M' = [id_{\mathbb{R}^3}]^{\mathcal{B}'}_{\mathcal{B}} = \,?$$

$$\begin{cases} -a + 2b + c = x_1 \\ -b = x_2 \\ a + 2c = x_3 \end{cases} \Leftrightarrow \begin{cases} b = -x_2 \\ -a + c = x_1 + 2x_2 \\ a + 2c = x_3 \end{cases}$$

$$\begin{cases} b = -x_2 \\ c = 1/3\,(x_1 + 2x_2 + x_3) \\ a = 1/3\,(-2x_1 - 4x_2 + x_3) \end{cases} \rightarrow M' = \begin{pmatrix} -10/3 & 5/3 & -2/3 \\ -2 & 1 & -1 \\ 5/3 & -1/3 & 4/3 \end{pmatrix}$$

Calcolando si verifica: $M \cdot M' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Esercizio 2 [non svolto in aula]

$$\begin{cases} -a + b = 3 \\ 2a + 12b = 1 \leftarrow \text{non serve} \\ a + 3b = -1 \end{cases} \quad \left\{ 2\cancel{4}b = \cancel{2} \right. \quad b = 1/2 \qquad a = -5/2$$

$$[v]_{\mathcal{B}} = \begin{pmatrix} -5/2 \\ 1/2 \end{pmatrix}$$

$$\begin{cases} 5a - b = x_1 \\ 4a + 9b = x_2 \\ -a + 3b = x_3 \end{cases} \to \begin{cases} a = 3b - x_3 \\ 15b - 5x_3 - b = x_1 \end{cases} \to b = (x_1 + 5x_3)/14 \qquad a = (3x_1 + x_3)/14$$

$$[v]_{\mathcal{B}'} = \begin{pmatrix} 4/7 \\ -1/7 \end{pmatrix}$$

$$\underset{\{v_1, v_2\}}{\mathcal{B}} = \mathcal{B}' \cdot M^{-1} \Rightarrow M^{-1} = \left([v_1]_{\mathcal{B}'}, [v_2]_{\mathcal{B}'} \right)$$

$$\Rightarrow M^{-1} = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 8/7 \end{pmatrix} \Rightarrow$$

$$M^{-1}[v]_{\mathcal{B}} = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 8/7 \end{pmatrix} \begin{pmatrix} -5/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 4/7 \\ -1/7 \end{pmatrix} = [v]_{\mathcal{B}'} \checkmark$$

Esercizio 3

$$f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}, \quad f\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{cases} 2a + b = 0 \\ a + b = 4 \\ -a = 4 \end{cases} \Longleftrightarrow \begin{cases} a = -4 \\ b = 8 \end{cases} \to \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = -4 \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + 8 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2a + b = -2 \\ a + b = 1 \\ -a = 3 \end{cases} \to \begin{cases} a = -3 \\ b = 4 \end{cases} \to \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -3 \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow [f]^{\mathcal{E}}_{\mathcal{B}} = \begin{pmatrix} -4 & -3 \\ 8 & 4 \end{pmatrix}, \qquad f \cdot \mathcal{B} = \mathcal{B}' \cdot [f]^{\mathcal{E}}_{\mathcal{B}}$$

$$f\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}-2\\9\\11\end{pmatrix},\quad f\begin{pmatrix}0\\2\\1\end{pmatrix}=\begin{pmatrix}2\\3\\1\end{pmatrix}$$

$$\begin{cases} b=-2\\ a+2b=9\\ a+b=11\end{cases}\to\begin{cases}a=13\\b=-2\end{cases}\qquad \begin{cases} b=2\\ a+2b=3\\ a+b=1\end{cases}\to\begin{cases}a=-1\\b=2\end{cases}$$

$$\to\quad [f]_{\mathcal{B}'}^{\mathcal{C}'}=\begin{pmatrix}13&-1\\-2&2\end{pmatrix}$$

$$\begin{cases}a+b=3\\a=2\end{cases}\to\begin{cases}a=2\\b=1\end{cases}\to\begin{pmatrix}3\\1\\2\end{pmatrix}=2\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\begin{cases}a+b=0\\a=1\end{cases}\to\begin{pmatrix}0\\2\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\Rightarrow\quad \mathcal{B}'=\mathcal{B}\cdot M=\mathcal{B}\cdot\begin{pmatrix}2&1\\1&-1\end{pmatrix}$$

$$\begin{cases}b=2\\a+b=-1\end{cases}\to\begin{cases}a=-3\\b=2\end{cases}\to\begin{pmatrix}2\\1\\-1\end{pmatrix}=-3\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}+2\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\begin{cases}b=1\\a+b=0\end{cases}\to\begin{matrix}a=-1\\b=1\end{matrix}\to\begin{pmatrix}1\\1\\0\end{pmatrix}=-\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\Rightarrow\quad \mathcal{C}=\mathcal{C}'N^{-1}=\mathcal{C}'\begin{pmatrix}-3&-1\\2&1\end{pmatrix}$$

$$N^{-1}[f]_{\mathcal{B}}^{\mathcal{C}}M=\begin{pmatrix}-3&-1\\2&1\end{pmatrix}\begin{pmatrix}-4&-3\\8&4\end{pmatrix}\begin{pmatrix}2&1\\1&-1\end{pmatrix}=$$

$$=\begin{pmatrix}-3&-1\\2&1\end{pmatrix}\begin{pmatrix}-11&-1\\20&4\end{pmatrix}=\begin{pmatrix}13&-1\\-2&2\end{pmatrix}=[f]_{\mathcal{B}'}^{\mathcal{C}'}\quad ✓$$

Esercizi 22/11/13

1. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \in \mathcal{M}_{3\times 2}(\mathbb{R})$.

$\mathcal{B} = \{v_1, v_2\}$. Dalla forma di $[f_A]^{\mathcal{E}_3}_{\mathcal{B}}$ vediamo che $Av_1 = 4A^1 + 7A^2 = 4A\begin{pmatrix}1\\0\end{pmatrix} + 7A\begin{pmatrix}0\\1\end{pmatrix} =$

$= A\begin{pmatrix}4\\7\end{pmatrix}$,

$Av_2 = A\begin{pmatrix}5\\-6\end{pmatrix} \Rightarrow \mathcal{B} = \left\{\begin{pmatrix}4\\7\end{pmatrix}, \begin{pmatrix}5\\-6\end{pmatrix}\right\}$.

2. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{R})$ $\quad \mathcal{B} = \{v_1, v_2\}$

$A\begin{pmatrix}1\\0\\0\end{pmatrix} = a_{11} e_1 + a_{21} e_2 = a_{11}\begin{bmatrix}-2\\3\end{bmatrix}_{\mathcal{B}} + a_{21}\begin{bmatrix}5\\-7\end{bmatrix}_{\mathcal{B}}$

$\begin{cases} e_1 = -2v_1 + 3v_2 \\ e_2 = 5v_1 - 7v_2 \end{cases}$ $\quad 5e_1 + 2e_2 = (15-14)v_2 = v_2$

$v_1 = \frac{1}{2}(-e_1 + 3v_2) = \frac{1}{2}(-e_1 + 15e_1 + 2e_2) = 7e_1 + e_2$

3. $[f]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$

$\begin{pmatrix}4\\1\end{pmatrix} = a\begin{pmatrix}2\\-1\end{pmatrix} + b\begin{pmatrix}3\\1\end{pmatrix} \xrightarrow{\text{ris. sistema}}$ $\quad a = 1/5$, $\quad b = 6/5$

$\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}17/5\\8/5\end{pmatrix} \Rightarrow$

$f\begin{pmatrix}4\\1\end{pmatrix} = \frac{17}{5}\begin{pmatrix}-1\\3\end{pmatrix} + \frac{8}{5}\begin{pmatrix}2\\1\end{pmatrix}$

$\mathcal{C} = \mathcal{E}' \cdot M \Rightarrow v = \mathcal{C} \cdot \begin{pmatrix}17/5\\8/5\end{pmatrix} = \mathcal{E}' \cdot M \cdot \begin{pmatrix}17/5\\8/5\end{pmatrix}$

$\Rightarrow$ dobbiamo trovare M:

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -3 \\ 2 \end{pmatrix} \xrightarrow{\text{risol.}} \begin{array}{l} a = 1 \\ b = 1 \end{array}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{bmatrix} a = 1 \\ b = 0 \end{bmatrix} \rightsquigarrow$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 17/5 \\ 8/5 \end{pmatrix} = \begin{pmatrix} 5 \\ 17/5 \end{pmatrix}$$

$\Rightarrow [f]_{\mathcal{B}'}^{\mathcal{E}'} = \begin{pmatrix} 5 & ? \\ 17/5 & ? \end{pmatrix}$. Calcoliamo le seconde colonne:

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = a \begin{pmatrix} 2 \\ -1 \end{pmatrix} + b \begin{pmatrix} 3 \\ 1 \end{pmatrix} \xrightarrow{\text{ris}} \begin{array}{l} a = 6/5 \\ b = 1/5 \end{array}$$

$$\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 6/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 13/5 \end{pmatrix}$$

$$M \cdot \begin{pmatrix} -3/5 \\ 13/5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3/5 \\ 13/5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3/5 \end{pmatrix}$$

$$\Rightarrow [f]_{\mathcal{B}'}^{\mathcal{E}'} = \begin{pmatrix} 5 & 2 \\ 17/5 & -3/5 \end{pmatrix}.$$

Esercizio 4

(a) $\det \begin{pmatrix} 3 & 4 \\ -5 & 7 \end{pmatrix} = 3 \cdot 7 - (-5 \cdot 4) = 41$

(b) $\det \begin{pmatrix} 1+t & -1+2t \\ 6+5t & 2-t \end{pmatrix} = (1+t)(2-t) - (6+5t)(-1+2t)$

$$= -11t^2 - 6t + 8$$

(c) $\det \begin{pmatrix} 3 & 4 & -1 \\ 2 & 5 & 1 \\ -3 & 4 & 2 \end{pmatrix} = 3 \cdot 5 \cdot 2 + 4 \cdot 1 \cdot (-3) + 2 \cdot 4 \cdot (-1)$

$$- (-3) \cdot 5 \cdot (-1) - 3 \cdot 4 \cdot 1 - 2 \cdot 4 \cdot 2$$

$$= -33$$

(d) $\det\begin{pmatrix} -1+t & 2 & -3 \\ 1+2t & 5 & -1 \\ -2 & 2-3t & 5 \end{pmatrix} =$

$= (-1+t)\cdot 5\cdot 5 + 2\cdot(-2)\cdot(-1) + (1+2t)(2-3t)(-3)$

$- (-2)\cdot 5\cdot(-3) - (2-3t)(-1)(-1+t) - (1+2t)\cdot 2\cdot(5)$

$= 15t^2 + 7t - 69.$

Esercizio 5.

(a) $A\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 26 \\ 17 \\ 2 \end{pmatrix}$, dove $A = \begin{pmatrix} 2 & -1 & 3 \\ -3 & 2 & 7 \\ 4 & 3 & -1 \end{pmatrix}$

$\det A = 2\cdot 2\cdot(-1) + (-1)\cdot 7\cdot 4 + (-3)\cdot 3\cdot 3 -$

$- 4\cdot 2\cdot 3 - 2\cdot 3\cdot 7 - (-1)(-3)(-1) =$

$= -122 \neq 0$

$\Rightarrow$ A invertibile $\Rightarrow$ il sistema ammette un'unica soluzione:

$\begin{cases} 2x - y + 3z = 26 \\ -3x + 2y + 7z = 17 \\ 4x + 3y - z = 2 \end{cases} \xrightarrow{\text{conti}} \begin{cases} x = 4 \\ y = -3 \\ z = 5 \end{cases}$

(b) $A = \begin{pmatrix} 4 & -3 & 7 \\ -3 & 5 & 1 \\ 10 & -13 & 5 \end{pmatrix}$. Calcolando si trova $\det A = 0$

$\Rightarrow f_A : \mathbb{R}^3 \to \mathbb{R}^3$ non è surgettiva, e ha $\operatorname{Ker} f \neq \{0\}$ $\Rightarrow$ il sistema ha $\begin{cases} 0 \\ \text{oppure} \\ \infty \end{cases}$ sol.

(7)

$$\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \begin{cases} 4x - 3y + 7z = 2 \\ -3x + 5y + z = -1 \\ 10x - 13y + 5z = 5 \end{cases}$$

$\mathrm{I} - 2 \cdot \mathrm{II}$:

$10x - 13y + 5z =$

$2 - (-2) = 4 \neq 5$

$\Rightarrow$ il sistema <u>non ammette</u> soluzioni.

(c) $A = \begin{pmatrix} 7 & -1 & 2 \\ 2 & 3 & -5 \\ 1 & -10 & 17 \end{pmatrix}$ $\det A = 0$.

$\begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -10 \end{pmatrix}$ indipendenti $\Rightarrow \dim \mathrm{Im}(f_A) = 2$.

$$\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = a \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} -1 \\ 3 \\ -10 \end{pmatrix} \iff$$

$$\begin{cases} 7a - b = 4 \\ 2a + 3b = -3 \\ a - 10b = 12 \end{cases}$$

$\left.\begin{matrix} 7a - b = 4 \\ 2a + 3b = -3 \end{matrix}\right\} \longrightarrow \begin{matrix} a = 9/23 \\ b = -29/23 \end{matrix}$

$9/23 - 10 \cdot \left(-\frac{29}{23}\right) = 13 \neq 12$

$\Rightarrow \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \notin \mathrm{Im}(f_A) \Rightarrow$ il sistema non ha soluzioni.