Esercizio 2(c) del 25/10/13

$V = \mathbb{R}_{\leq 3}[t]$, v_1, v_2 e v_3 sono lin. indip. perché di gradi diversi.

$\dim V = 4 \Rightarrow$ dobbiamo aggiungere un altro vettore.

Ma $\quad v_3 + v_2 + v_1 = v_4$, e

$\quad -v_3 + 5v_2 + v_1 = v_5$, quindi

v_4 e v_5 vanno scartati.

$v_6 = 2 - t + 3t^2 - 4t^3 \stackrel{?}{=} a v_1 + b v_2 + c v_3 =$

$= (b+5c) + (-a-b+2c)t + bt^2 - ct^3$

$\Rightarrow c = 4,\ b = 3,$ ma $3 + 5 \cdot 4 \neq 2 \Rightarrow$

$v_6 \notin \mathrm{Span}(v_1, v_2, v_3)$

$\Longrightarrow \mathcal{B} = \{v_1, v_2, v_3, v_6\}$ base di V.

(a) $W = \mathrm{Span}\left(\begin{pmatrix}-1\\4\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\\3\end{pmatrix}\right), \quad Z = \mathrm{Span}\left(\begin{pmatrix}3\\2\\1\\-1\end{pmatrix}, \begin{pmatrix}-9\\3\\0\\8\end{pmatrix}\right) \leq \mathbb{R}^4$

$\dim W = 2$ perché i generatori non sono proporzionali. $\dim Z = 2$ per la stessa ragione.

$$W \cap Z : \begin{cases} -a + 2b = 3c - 9d \\ 4a + b = 2c + 3d \\ 2a - b = c \\ a + 3b = -c + 8d \end{cases}$$

$$\begin{cases} c = 2a - b \\ -a + 2b = 6a - 3b - 9d \\ \cancel{4a} + b = \cancel{4a} - 2b + 3d \\ a + 3b = -2a + b + 8d \end{cases} \iff$$

$$\begin{cases} c = 2a - b \\ b = d \\ 7a - 5b - 9d = 0 \\ 3a + 2b - 8d = 0 \end{cases} \iff \begin{cases} d = b \\ c = 2a - b \\ \cancel{7}a = {}^{2}\cancel{14}\, b \\ \cancel{3}a = {}^{2}\cancel{6}\, b \end{cases}$$

$$\Rightarrow W \cap Z = \left\{ \begin{pmatrix}2b\\b\\3b\\b\end{pmatrix} : b \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{pmatrix}2\\1\\3\\1\end{pmatrix}\right)$$

$$W + Z : \begin{pmatrix}-1\\4\\2\\1\end{pmatrix} = a\begin{pmatrix}3\\2\\1\\-1\end{pmatrix} + b\begin{pmatrix}-9\\3\\0\\8\end{pmatrix} = \begin{pmatrix}3a - 9b\\2a + 3b\\a\\-a + 8b\end{pmatrix}$$

$$\Rightarrow \begin{array}{l} a = 2 \\ 8b = 3 \\ 3b = 0 \end{array}$$

nessuna soluzione.

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$$\begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} = a \begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} -9 \\ 3 \\ 0 \\ 8 \end{pmatrix} + c \begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{cases} 3a - 9b - c = 2 \\ 2a + 3b + 4c = 1 \\ a \quad + 2c = -1 \\ -a + 8b + c = 3 \end{cases} \rightarrow \begin{cases} a = -1 - 2c \\ -3 - 6c - 9b - c = 2 \\ -2 - \cancel{4c} + 3b + \cancel{4c} = 1 \\ 1 + 2c + 8b + c = 3 \end{cases}$$

$$\begin{cases} a = -1 - 2c \\ -9b - 7c = 5 \\ b = 1 \\ 3c = -6 \end{cases} \quad \begin{cases} b = 1 \\ c = -2 \\ a = 3 \\ -9 \cdot 1 - 7 \cdot (-2) = 5 \end{cases}$$

✓

OK

$\Rightarrow \quad \dim W = \dim Z = 2, \ \dim(Z + W) = 3$

$\dim(Z \cap W) = 1$

(e) $Z : p(-1) = p''(\frac{1}{6}) = 0 \qquad V = \mathbb{R}_{\leq 4}[t]$

$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$

$p'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3$

$p''(t) = 2a_2 + 6a_3 t + 12 a_4 t^2$

$$\begin{cases} a_0 - a_1 + a_2 - a_3 + a_4 = 0 \\ 2a_2 + a_3 + \frac{1}{3} a_4 = 0 \end{cases}$$

$$\begin{cases} a_4 = -6a_2 - 3a_3 \\ a_0 = a_1 + 5a_2 + 4a_3 \end{cases} \Rightarrow \mathcal{B} = \begin{Bmatrix} 1 + t \\ 5 + t^2 - 6t^4 \\ 4 + t^3 - 3t^4 \end{Bmatrix}$$

$1 - t + t^2 + t^3 \in Z$? NO

$1 - (-1) + (-1)^2 + (-1)^3 = +1 + 1 + 1 - 1 = 2 \neq 0$

$$Z + W = \mathrm{Span}\begin{pmatrix} 1+t,\ 5+t^2-6t^4, \\ 4+t^3-3t^4,\ 1-t+t^2+t^3, \\ 2+t-2t^3+3t^4 \end{pmatrix}$$

$2 + t - 2t^3 + 3t^4 \in$ Span(primi quattro) ?

$$\begin{cases} a + 5b + 4c + d = 2 \\ a - d = 1 \\ b + d = 0 \\ c + d = -2 \\ -6b - 3c = 3 \end{cases}$$

$$\begin{cases} b = -d \\ a = d + 1 \\ c = -d - 2 \\ d + 1 - 5d - 4d - 8 + d = 2 \iff d = -9/7 \\ 6d + 3d + 6 = 3 \iff d = -1/3 \end{cases}$$

$\Rightarrow Z + W = \mathbb{R}_{\leq 4}[t]$

$Z \cap W$: $W = \{ a + 2b + (-a+b)t + at^2 + (a-2b)t^3 + 3bt^4 \}$

$p(-1) = a + 2b + a - b + \cancel{a} - \cancel{a} + 2b + 3b$

$p''(\frac{1}{6}) = 2a + a - 2b + \cancel{36}\, b \frac{1}{\cancel{36}} = 0$

$$\begin{cases} 2a + 6b = 0 \\ 3a - b = 0 \end{cases} \Rightarrow a = b = 0$$

$\Rightarrow W \cap Z = \{0\}$

Es. 1/11/13 (a) Si verifica immediatamente che $x \mapsto x_1 v_1 + x_2 v_2 + x_3 v_3$ è lineare

$$f(x) = x_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} -4 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 7 \end{pmatrix} \Rightarrow f \text{ lineare},$$

$$\mathrm{Im}(f) = \mathrm{Span}\left(\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \end{pmatrix} \right) = \mathbb{R}^2$$

perché $\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ sono lin. indipendenti.

$f(x) = 0,\ x \neq 0 \Rightarrow x_3 \neq 0$, altrimenti $\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ dipendenti.

$\Rightarrow$ se $f(x) = 0,\ x \neq 0$, abbiamo

$$\begin{pmatrix} 5 \\ 7 \end{pmatrix} = -\frac{x_1}{x_3} \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \frac{x_2}{x_3} \begin{pmatrix} -4 \\ 5 \end{pmatrix}.$$

$$\begin{cases} 3a - 4b = 5 \\ -2a + 5b = 7 \end{cases} \overset{\text{conti}}{\rightsquigarrow} \begin{cases} a = 53/7 \\ b = 31/7 \end{cases}$$

$$\Rightarrow x_1 = -\frac{53}{7} x_3, \quad x_2 = -\frac{31}{7} x_3$$

$$\mathrm{Ker} f = \mathrm{Span}\left(\begin{pmatrix} -53/7 \\ -31/7 \\ 1 \end{pmatrix} \right) = \mathrm{Span}\left(\begin{pmatrix} -53 \\ -31 \\ 7 \end{pmatrix} \right)$$

$$\Rightarrow \dim \mathrm{Ker} f + \dim \mathrm{Im} f = 2 + 1 = 3 = \dim \mathbb{R}^3 \quad \checkmark$$

(e) $f(x) = x_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$ è lineare come in (a)

$$\mathrm{Im} f = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} \right)$$

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I primi due vettori sono indipendenti.
E tutti e tre? Impostiamo il sistema:

$$\begin{cases} a + b = 0 \\ b + 2c = 0 \\ 2a - c = 0 \\ a + b + c = 0 \end{cases}$$

$a = -b = 2c$

$b = -2c$

$a = c/2 \longrightarrow c = 0 = a = b$

$\Rightarrow$ $\dim Im(f) = 3$. Lo stesso conto dimostra $Ker(f) = \{0\} \Rightarrow \dim Ker(f) = 0$

$\Rightarrow 0 + 3 = 3 = \dim \mathbb{R}^3$ ✓