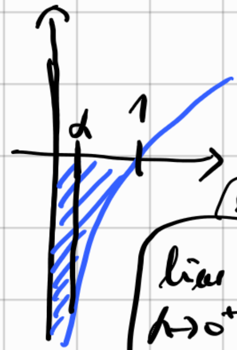
ANALISI MATEMATICA B
LEZIONE 50 - 7.2.2024

Integrali impropri

Esempio 1 $\displaystyle\int_0^1 \ln x \, dx$



per definizione $\overset{\text{DEF}}{=} \displaystyle\lim_{\alpha\to 0^+} \int_\alpha^1 \ln x \, dx$

$$= \lim_{\alpha\to 0^+} \left[x\ln x - x \right]_\alpha^1$$

$$= \lim_{\alpha\to 0^+} \left[-1 - (\underbrace{\alpha \ln \alpha}_{\to 0} - \underbrace{\alpha}_{\to 0}) \right]$$

$$= -1$$

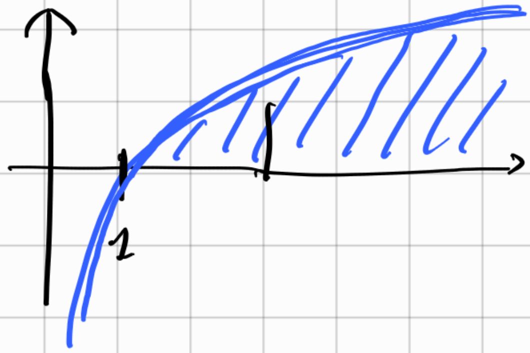
MONOLATERALE

RICORDIAMO

$$\lim_{\alpha\to 0^+} \alpha\ln\alpha = \lim_{t\to+\infty} \frac{-\ln t}{t} = 0 \qquad t = \frac{1}{\alpha}$$

$$\ln t << t^\alpha$$

ES 2 $\displaystyle\int_1^{+\infty} \ln x \, dx =$



$$\overset{\text{DEF}}{=} \lim_{\beta\to+\infty} \int_1^\beta \ln x \, dx$$

$$= \lim_{\beta\to+\infty} \left[x\ln x - x \right]_1^\beta = \lim_{\beta\to+\infty} \beta\ln\beta - \beta - (-1)$$

$$= +\infty$$

MONOLATERALE

ES 3 $\displaystyle\int_0^{+\infty} \ln x \, dx \overset{\text{DEF}}{=} \int_0^c \ln x \, dx + \int_c^{+\infty} \ln x \, dx$

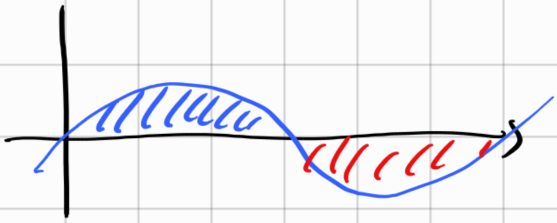
per definizione

BILATERALE

$$= \lim_{\alpha\to 0^+} \left[x\ln x - x \right]_\alpha^c + \lim_{\beta\to+\infty} \left[x\ln x - x \right]_c^\beta$$

$$= (c\ln c - c) - (0 - 0) + (+\infty) - (c\ln c - c) = +\infty$$

ES 4 $\displaystyle\int_0^{+\infty} \sin x \, dx$

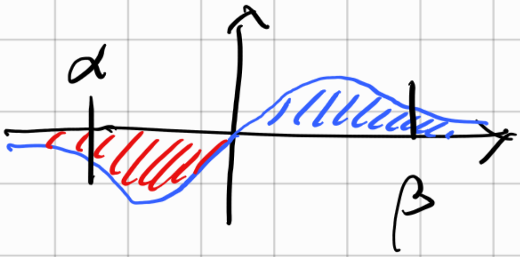


$$\overset{\text{DEF}}{=} \lim_{\beta \to +\infty} \int_0^{\beta} \sin x \, dx$$

$$= \lim_{\beta \to +\infty} \Big[-\cos x\Big]_0^{\beta} = \lim_{\beta \to +\infty} \left[-\cos\beta + 1\right]$$

il limite non esiste

ES 5 $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{1+x^2} \, dx$



$$\overset{\text{DEF}}{=} \int_{-\infty}^{0} \frac{x}{1+x^2} \, dx + \int_0^{+\infty} \frac{x}{1+x^2} \, dx$$

$$= \lim_{\alpha \to -\infty} \int_{\alpha}^{0} \frac{x}{1+x^2} \, dx + \lim_{\beta \to +\infty} \int_0^{\beta} \frac{x}{1+x^2} \, dx$$

$$= \lim_{\alpha \to -\infty} \left[\frac{1}{2} \ln(1+x^2)\right]_{\alpha}^{0} + \lim_{\beta \to +\infty} \left[\frac{1}{2} \ln(1+x^2)\right]_0^{\beta}$$

$$= \lim_{\alpha \to -\infty} -\frac{1}{2} \ln(1+\alpha^2) + \lim_{\beta \to +\infty} \frac{1}{2} \ln(1+\beta^2)$$

$$= -\infty + (+\infty) \qquad \underline{\text{non esiste}}$$

NON È DEFINITO.

⚠ $$\lim_{\beta \to +\infty} \int_{-\beta}^{\beta} \frac{x}{1+x^2} \, dx = 0$$

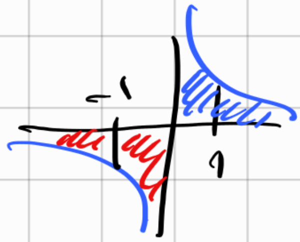
Integrale improprio multilaterale.

Se $f: I \setminus \{x_1, \dots, x_N\} \to \mathbb{R}$ $\quad a = \inf I$, $\quad b = \sup I$

$$\int_a^b f(x)\,dx \overset{\text{DEF}}{=} \int_a^{x_1} f + \int_{x_1}^{x_2} f + \dots + \int_{x_N}^{b} f$$

la somma deve avere senso: tutti gli eventuali infiniti devono avere lo stesso segno.

ES $\quad \displaystyle\int_{-\infty}^{+\infty} \frac{1}{x}\,dx$ non esiste



$$\int_{-\infty}^{+\infty} \frac{1}{x} = \underbrace{\int_{-\infty}^{-1} \frac{1}{x}\,dx}_{= -\infty} + \underbrace{\int_{-1}^{0} \frac{1}{x}\,dx}_{= -\infty} + \underbrace{\int_{0}^{1} \frac{1}{x}\,dx}_{= +\infty} + \underbrace{\int_{1}^{+\infty} \frac{1}{x}\,dx}_{= +\infty}$$

$$\int \frac{1}{x} = \ln|x|.$$

NOTAZIONE

$$[F]_a^b = \lim_{\beta \to b^-} F(\beta) - \lim_{\alpha \to a^+} F(\alpha)$$

precedentemente: $\quad [F]_a^b = F(b) - F(a)$

Se F è continua sono la stessa cosa.

OSS

$$b \longmapsto \int_a^b f(x)\,dx \quad \text{è continua}$$

$$a \longmapsto \int_a^b f(x)\,dx \quad \text{è continua.}$$

se f è limitata e R-int.

$$\int_a^{b+\varepsilon} f - \int_a^b f = \int_b^{b+\varepsilon} f \le \varepsilon \cdot \sup|f| \longrightarrow 0.$$

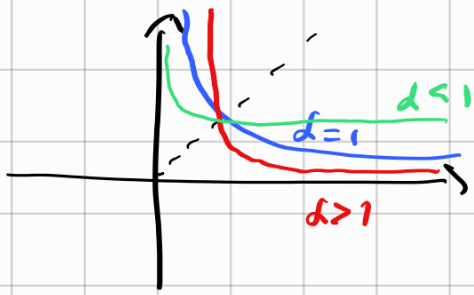
Con la notazione:

$1-\alpha>0$

$$\int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \left[\dfrac{x^{1-\alpha}}{1-\alpha}\right]_0^1 = \begin{cases} \dfrac{1}{1-\alpha} - 0 & \text{se } \alpha<1 \\ +\infty & \text{se } \alpha>1 \end{cases} \\ \left[\ln x\right]_0^1 = +\infty & \alpha = 1 \end{cases}$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \left[\dfrac{x^{1-\alpha}}{1-\alpha}\right]_1^{+\infty} = \begin{cases} +\infty & \text{se } \alpha<1 \\ 0 - \dfrac{1}{1-\alpha} = \dfrac{1}{\alpha-1} & \text{se } \alpha>1 \end{cases} \\ \left[\ln x\right]_1^{+\infty} = +\infty & \alpha = 1 \end{cases}$$

assomiglia a $\displaystyle\sum_{k\ge1}^{\infty} \frac{1}{k^\alpha}$



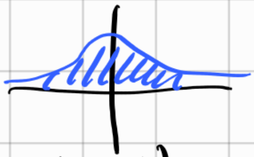
OBIETTIVO. 1. Se so calcolare l'integrale, voglio calcolarlo.

2. Se non so calcolarlo voglio almeno sapere se converge oppure no.

ES $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$ converge o diverge?

STRUMENTI PER DETERMINARE IL CARATTERE

1. Confronto / Monotonia $\qquad f \le g \Rightarrow \int_a^b f \le \int_a^b g$
(se f,g hanno integrale)



ES $0 \le \int_1^{+\infty} e^{-x^2}\,dx \le \int_1^{+\infty} e^{-x}\,dx = \left[-e^{-x}\right]_1^{+\infty} = 0 - (-e^{-1}) = \frac{1}{e} < +\infty$

$\downarrow$

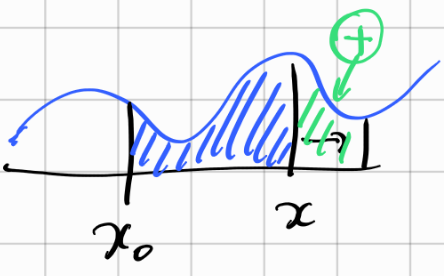
Se ha integrale, l'integrale è finito

0. Se $f \geq 0$ e f loc. R-integrabile, allora $\int_a^b f$ esiste sempre ≥ 0 $(a \le b)$

Teorema. Dim. $\qquad F(x) = \int_{x_0}^{x} f$ è crescente se $f \geq 0$

Se $x_2 \geq x_1$ $\quad F(x_2) - F(x_1) = \int_{x_1}^{x_2} f \geq 0$

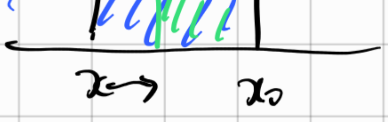
$\Rightarrow F(x_2) \geq F(x_1)$



$\lim_{\beta \to b^-} \int_{x_0}^{\beta} f = \lim_{\beta \to b^-} F(\beta)$ esiste ≥ 0 essendo F monotona.

$\lim_{\alpha \to a^+} \int_{\alpha}^{x_0} f = \lim_{\alpha \to a^+} G(\alpha)$ esiste ≥ 0

$G(x) = \int_x^{x_0} f$ è decrescente se $f \geq 0$



$\square$

2. Confronto asintotico

ⓐ $\int_a^b f$ impropio in a (monolaterale)

$\frac{f}{g} \to 1$

se $f \sim g$ per $x \to a^+$, f loc. R. int.
g loc. R. int.
$f, g \geq 0$

allora $\int_a^b f$ e $\int_a^b g$

hanno lo stesso carattere. (Analogamente in b)

ES $\int_1^{+\infty} \frac{1}{x+\sin x} \, dx$

$\frac{1}{x+\sin x} \sim \frac{1}{x}$ per $x \to +\infty$

definitivamente positiva.

$\int_1^{+\infty} \frac{1}{x} = +\infty$ dunque anche l'integrale dato è divergente

(b) Se $f \ll g$ per $x \to a^+$ $f \geq 0$
$g \geq 0$
loc. R. integrab.

$\frac{f}{g} \to 0$

Se $\int_a^b g < +\infty \Rightarrow \int_a^b f < +\infty$

Se $\int_a^b f = +\infty \Rightarrow \int_a^b g = +\infty$

ES $e^{-x^2} \ll \frac{1}{x^2}$ $\int_1^{+\infty} e^{-x^2}$ è convergente

perché $\int_1^{+\infty} \frac{1}{x^2} \, dx$ lo è.

Esempio

$$\int_0^1 \frac{1}{x-\sin x}\,dx \quad \text{converge?}$$

$$\sin x = x - \frac{x^3}{6} + o(x^3) \quad \text{per } x \to 0$$

$$x - \sin x = \frac{x^3}{6} + o(x^3) \quad \text{per } x \to 0$$

$$x - \sin x \sim \frac{x^3}{6} \qquad \frac{1}{x-\sin x} \sim \frac{6}{x^3}$$

$$\int_0^1 \frac{6}{x^3}\,dx = 6\int_0^1 \frac{1}{x^3}\,dx = +\infty$$

Anche l'integrale dato diverge.

Esempio

$$\int_0^1 \frac{1}{(x-\sin x)^\alpha}\,dx$$

Per quali $\alpha \in \mathbb{R}$, l'integrale converge? $\left[\alpha < \frac{1}{3}\right]$