ANALISI MATEMATICA B
LEZIONE 74 - 1.4.2022

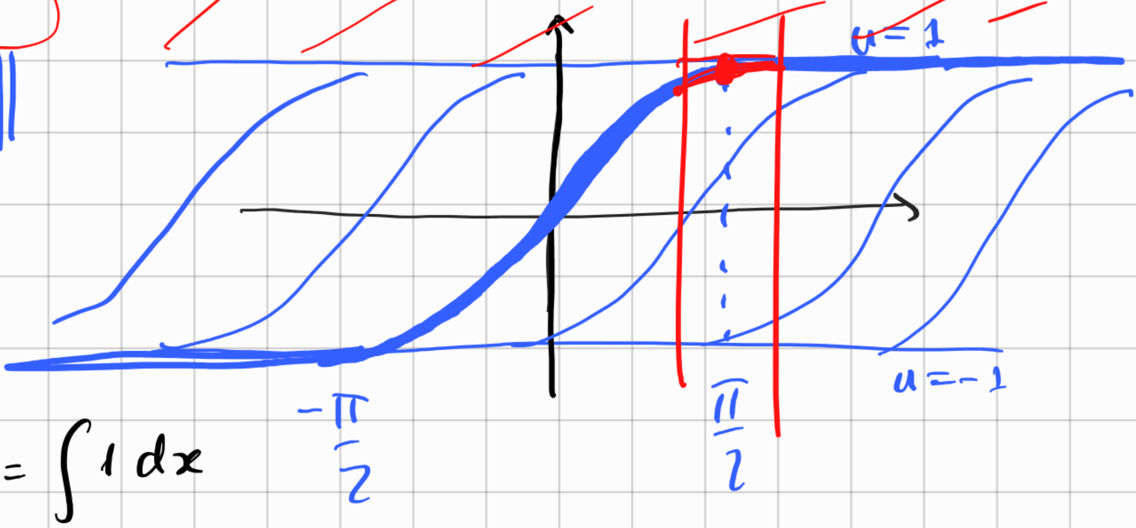
Esercizio

$$\begin{cases} u'(x) = \sqrt{1-u^2(x)} \\ u(0) = 0 \end{cases} \quad (*)$$

$u(\frac{\pi}{2}) = 1$

$u' = \sqrt{1-u^2}$

$\sqrt{1-u^2} \overset{?}{=} 0$ se $u = \pm 1$.

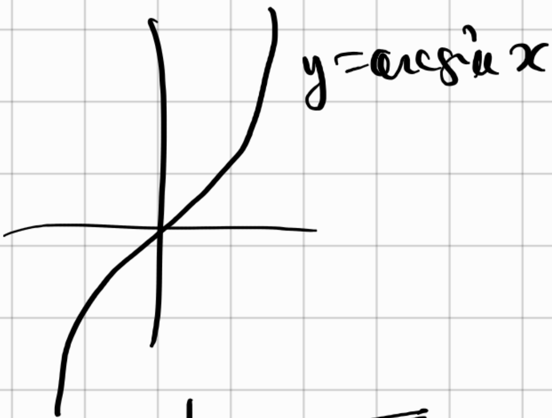


Dove $u(x) \neq \pm 1$ posso dividere:

$$\frac{u'}{\sqrt{1-u^2}} = 1$$

$$\int \frac{u'(x)}{\sqrt{1-u^2(x)}}\,dx = \int 1\,dx$$

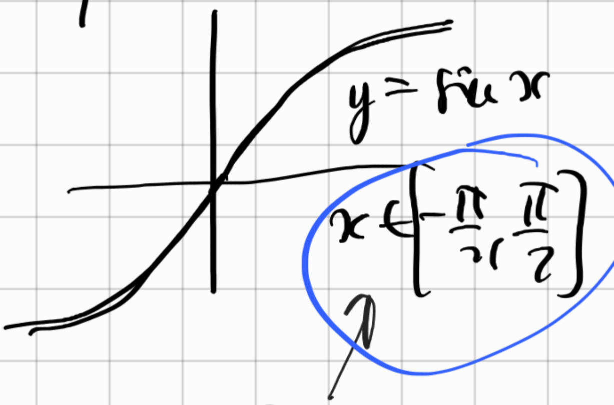
$$\int \frac{1}{\sqrt{1-u^2}}\,du = x + c$$



$$\arcsin u = x + c$$

$$u = \sin(x+c)$$

$u(0) = 0 \Rightarrow c = 0$

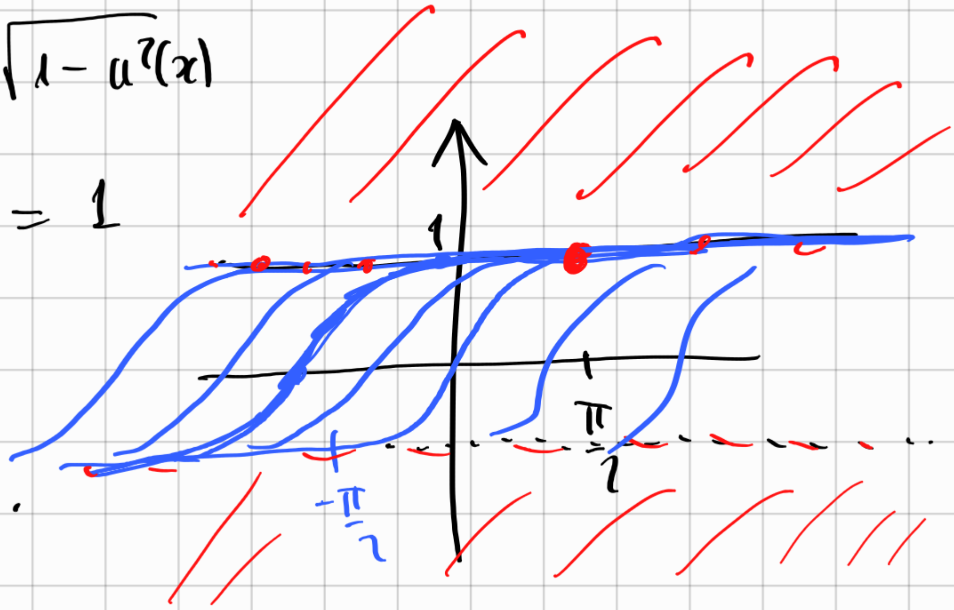


Però questa soluzione può essere estesa:

$$u(x) = \begin{cases} 1 & \text{se } x > \frac{\pi}{2} \\ \sin x & \text{se } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -1 & \text{se } x < -\frac{\pi}{2} \end{cases}$$

Osservazione

$$\begin{cases} u'(x) = \sqrt{1-u^2(x)} \\ u\left(\frac{\pi}{2}\right) = 1 \end{cases}$$



$u(x) = 1$ è soluzione.

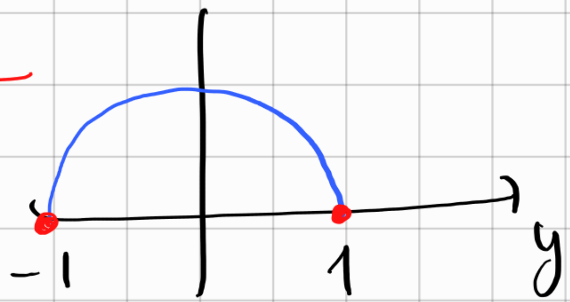
Non è l'unica!

$\forall c \geq 0$

$$u_c(x) = \begin{cases} 1 & \text{se } x > \frac{\pi}{2} - c \\ \sin(x+c) & \text{per } x \in \left[-\frac{\pi}{2} - c, \frac{\pi}{2} - c\right] \\ -1 & \text{se } x < -\frac{\pi}{2} - c \end{cases}$$

$$u' = f(u(x)) \qquad \boxed{f \in C^1}$$

$$f(y) = \sqrt{1-y^2}$$



$$u' = f(u(x), x)$$

$$\begin{cases} u' = \sqrt{u} \\ u(0) = 0 \end{cases}$$ Baffo di Peano $$\begin{cases} u' = \sqrt[3]{u} \\ u(0) = 0 \end{cases}$$

Eq. diff. ord. lineari coeff. cost. di ordine n. non omogenee

$a_0, a_1 \dots a_n \in \mathbb{R}$

$$u^{(n)} + a_{n-1} u^{(n-1)} + \dots + a_1 u' + a_0 u = g(x) \qquad (*)$$

Risolvo l'omogenea: $u_0(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \dots + C_n e^{\lambda_n x}$

Se u_* è una sol. particolare di $(*)$ tutte le sol. di $(*)$ sono: $u(x) = u_*(x) + \boxed{u_0(x)}$

Metodo di somiglianza:

$u'' + u = 0$ $\qquad u(x) = c_1 \sin x + c_2 \cos x$

$P(\lambda) = \lambda^2 + 1$ $\qquad \lambda_{12} = \pm i$

$u'' + u = \sin x$ $\qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$\lambda_1 = i \quad \lambda_2 = -i$

$m = 1$

$\mu = \pm i \leftarrow (D-i)(D+i)$

$u_*(x) = a \cdot x \cdot \cos x + b \cdot x \cdot \sin x$

$u_*'(x) = a \cos x - a x \sin x + b \sin x + b x \cos x$

$u_*''(x) = -a \sin x - a \sin x - a x \cos x + b \cos x + b \cos x - b x \sin x$

$= -2a \sin x - a x \cos x + 2b \cos x - b x \sin x$

$u_*'' + u_* = -2a \sin x + 2b \cos x \overset{!}{=} \sin x.$

$$\begin{cases} -2a = 1 \\ 2b = 0 \end{cases} \qquad \begin{cases} a = -\frac{1}{2} \\ b = 0 \end{cases}$$

$$u_p(x) = -\frac{1}{2}\, x \cos x$$

$$u(x) = -\frac{1}{2} x \cos x + c_1 \sin x + c_2 \cos x$$

$$= \left(c_2 - \frac{1}{2} x\right) \cos x + c_1 \sin x$$

Es

$$u'' + 2u' + u = \frac{x}{e^x}$$

$$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \qquad \lambda_{1,2} = -1 \qquad m = 2.$$

Sol. omogenea:
$$u_0(x) = c_1 e^{-x} + c_2 x e^{-x} = (c_1 + c_2 x) e^{-x}$$

Sol. particolare non omogenea: $\quad g(x) = x e^{-x}$ $\quad \mu = -1$, $m = 2$

$2 = m$

$1 \cdot \quad u_p(x) = (ax+b)\, x^2 e^{-x} = (ax^3 + bx^2) e^{-x}$

stesso grado

$+$

$2 \cdot \quad u_p'(x) = (3ax^2 + 2bx - ax^3 - bx^2)\, e^{-x}$

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$1 \cdot \quad u_p''(x) = (6ax + 2b - 3ax^2 - 2bx - 3ax^2 - 2bx + ax^3 + bx^2)\, e^{-x}$

$$u_*'' + 2u_*' + u_* = \left(6ax + 2b - \cancel{4bx} + \cancel{4bx}\right) e^{-x} \overset{!}{=} xe^{-x}$$

$$\begin{cases} 6a = 1 \\ 2b = 0 \end{cases} \qquad \begin{cases} a = \frac{1}{6} \\ b = 0 \end{cases}$$

$$u_*(x) = \frac{1}{6}x^3 e^{-x}$$

Tutte le sol:

$$u(x) = \frac{1}{6}x^3 e^{-x} + (c_1 + c_2 x) e^{-x}$$

$$= \left(c_1 + c_2 x + \frac{1}{6}x^3\right) e^{-x} \, .$$

Metodo della variazione delle costanti

Es

$$u'' - 2u' = e^x \sin x$$

$$P(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2) \quad \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 2 \end{cases}$$

Sol. eq. omogenea associata $\quad u'' - 2u' = 0$

$$u_0(x) = \boxed{c_1 + c_2 e^{2x}}$$

Cerco u_* sol. particolare della non omogenea:

$$(e^{0x})'$$

$$u_*(x) = c_1(x) + \underline{c_2(x) e^{2x}}$$

$$u_*'(x) = \cancel{c_1'(x)} + \cancel{c_2'(x) e^{2x}} + \underbrace{0 + 2c_2(x) e^{2x}}$$

$$\to \quad u_*''(x) = \underline{0 + 2c_2'(x) e^{2x}} + 0 + 4c_2(x) e^{2x}$$

$$\begin{cases} c_1'(x) + c_2'(x)e^{2x} = 0 \\ 2c_2'(x)e^{2x} = e^x \sin x \end{cases}$$

$$\left.\begin{array}{l} u_y''(x) \\ -2u_y'(x) \end{array}\right] = \left[\begin{array}{l} \boxed{0 + 2c_2'(x)e^{2x}} + \boxed{0 + 4c_2(x)e^{2x}} \\ -2\left[\boxed{\cancel{0} + \cancel{0}}\right] + 0 + 2c_2(x)e^{2x} \end{array}\right.$$

$$= 0$$

$$u_y'' - 2u_y' = 2c_2'(x)e^{2x} \overset{!}{=} e^x \sin x$$

$$\begin{pmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{pmatrix} \cdot \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^x \sin x \end{pmatrix}$$

$$\begin{cases} c_2'(x) = \dfrac{e^x \sin x}{2e^{2x}} = \dfrac{1}{2} e^{-x} \sin x \\ c_1'(x) + \dfrac{1}{2} e^{-x} \sin x \, e^{2x} = 0 \end{cases}$$

$$\begin{cases} c_2'(x) = \dfrac{1}{2} e^x \sin x \\ c_1'(x) = -\dfrac{1}{2} e^x \sin x \end{cases}$$

$$c_1(x) = \frac{1}{2}\int e^x \sin x \, dx = \frac{1}{4} e^x (\sin x - \cos x)$$

$$c_2(x) = -\frac{1}{4} e^x (\sin x - \cos x)$$

$$u_p(x) = \frac{1}{4} e^x (\sin x - \cos x) - \frac{1}{4} e^x (\sin x - \cos x)\, e^{2x}$$

$$= \frac{1}{4} e^x (\sin x - \cos x) - \frac{1}{4} e^{3x} (\sin x - \cos x)$$

Sol. generale:

$$u(x) = u_p(x) + c_1 + c_2 e^{2x} \qquad \square$$

$$u'' + u = \frac{1}{\cos x}$$