ANALISI MATEMATICA B
LEZIONE 68 - 18.3.2022

Integrali impropri

$\int_a^b f(x)\,dx$ integrale improprio

se $f(x) \geq 0 \quad \forall x$

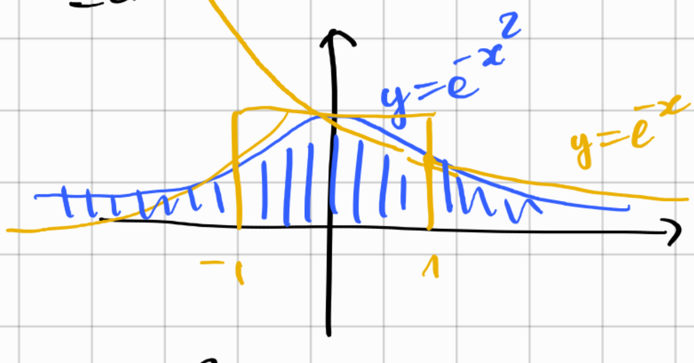
l'integrale esiste. finito / infinito

Criterio di confronto

• confronto puntuale.

se $f \leq g \Rightarrow \int_a^b f \leq \int_a^b g$

Es $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$ esiste finito?



$e^{-x^2} \leq e^{-x}$ se $x>1$

$$\int_1^{+\infty} e^{-x^2}\,dx \leq \int_1^{+\infty} e^{-x}\,dx = \left[-e^{-x}\right]_1^{+\infty} = 0 + \frac{1}{e}$$

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \underbrace{\int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{+\infty} e^{-x^2}\,dx}_{=\text{ finiti}}$$

- confronto asintotico $\quad f>0,\ g>0$

$$f,g:[a,b)\to\mathbb{R}$$

↑ punto cattivo

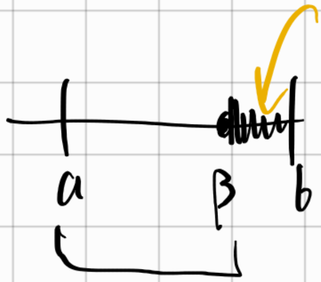
$f << g$ per $x\to b^-$ $\qquad \lim_{x\to b^-}\frac{f(x)}{g(x)}=0$

Allora se $\int_a^b g<+\infty$ anche $\int_a^b f<+\infty$.

<u>dim</u> in un intorno di b $\quad \frac{f}{g}<1$.

$$\Downarrow$$

$$f<g$$



<u>Es</u>. $\int_0^{+\infty}\frac{\ln x}{x^2+\sqrt{x}}\,dx$ converge?

Punti "cattivi" sono $+\infty$, 0 $\qquad (0,+\infty)$

$$\int_0^{+\infty}=\int_0^1+\int_1^{+\infty}$$

(2) $\quad$ (1)

$$\int_1^{+\infty}\frac{1}{x^p}\,dx<+\infty \iff p>1$$

(1) $\underbrace{\int_1^{+\infty}\frac{\ln x}{x^2+\sqrt{x}}\,dx}\in\mathbb{R}$

↑ esiste finito

$$\frac{\ln x}{x^2+\sqrt{x}} \overset{!}{<<} \frac{1}{x^{3/2}} \quad \text{per } x\to+\infty$$

↑

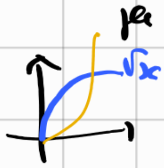
$$\frac{\frac{\ln x}{x^2+\sqrt{x}}}{\frac{1}{x^{3/2}}}=\frac{x^{3/2}\cdot\ln x}{x^2+\sqrt{x}}=\frac{x^{3/2}}{x^2}\,\frac{\ln x}{1+\frac{\sqrt{x}}{x^2}}\to 0$$

$$\frac{1}{\sqrt{x}}=\frac{x^{3/2}}{x^2}$$

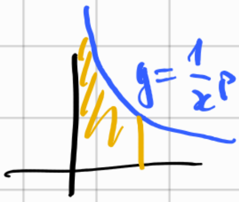
(2) $\int_0^1 \frac{-\ln x}{x^2+\sqrt{x}}\,dx$ positiva

$$\frac{-\ln x}{x^2+\sqrt{x}}$$ per $x \to 0$

$\sqrt{x} \gg x^2$ per $x \to 0$



$$\int_0^1 \frac{1}{x^p}\,dx < +\infty \quad \Updownarrow \quad p<1.$$



per $x \to 0^+$

$$\boxed{\frac{-\ln x}{x^2+\sqrt{x}}} \ll \boxed{\frac{1}{x^{2/3}}}$$

$$\frac{\frac{-\ln x}{x^2+\sqrt{x}}}{\frac{1}{x^{2/3}}} = \frac{-x^{2/3}\cdot \ln x}{x^2+\sqrt{x}} = \underbrace{\frac{x^{2/3}}{\sqrt{x}}\cdot \frac{-\ln x}{\frac{x^2}{\sqrt{x}}+1}}_{x^{1/6} \,\to 0} \longrightarrow 0$$

$\frac{x^2}{\sqrt{x}} = x\sqrt{x} \to 0$

$\int_0^1 \frac{-\ln x}{x^2+\sqrt{x}}\,dx$ esiste finito positivo

(2) $\int_0^1 \frac{\ln x}{x^2+\sqrt{x}}\,dx$ esiste finito negativo.

(1)+(2) $\int_0^{+\infty} \frac{\ln x}{x^2+\sqrt{x}}\,dx$ esiste finito.

- Se $f \sim g$ (cioè $\frac{f}{g} \to 1$) per $x \to b$

$f,g : [a,b) \to \mathbb{R}$ $\qquad f, g > 0$

$\int_a^b f$ e $\int_a^b g$ hanno lo stesso carattere

dim

se $\frac{f}{g} \to 1$ in un intorno di b

$\frac{1}{2} g \le f \le 2g$

Es $\int_0^{+\infty} \frac{1}{x^2+\ln x}\,dx$

$+\infty$ è cattivo

0 è cattivo? sì e no. $f(x) = \frac{1}{x^2+\ln x}$ si può estendere con continuità in $x=0$

① $= \int_0^{\varepsilon} f(x)\,dx$ esiste finito

$\lim_{x\to 0^+} f(x) = 0$

② $= \int_1^{+\infty} \frac{1}{x^2+\ln x}\,dx$ converge?

per $x \to +\infty$

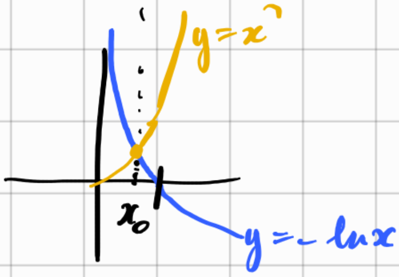
$x^2 + \ln x \sim x^2$

$\frac{1}{x^2+\ln x} \sim \frac{1}{x^2}$

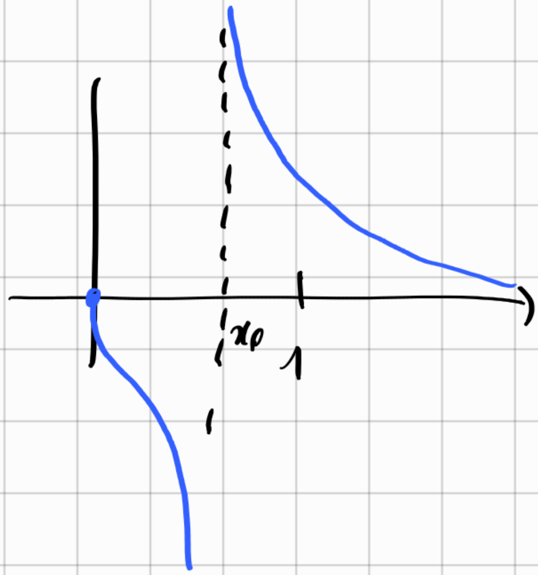
$\int_1^{+\infty} \frac{1}{x^2}\,dx < +\infty$ anche ② converge.

Attenzione $x^2+\ln x = 0$

$x^2 = -\ln x$



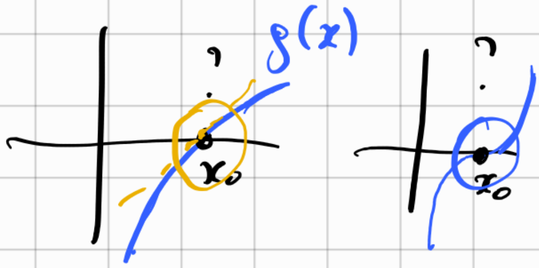
$\exists!\ x_0 \in (0,1)$ t.c. $x_0^2 + \ln x_0 = 0$



$g(x) = \boxed{x^2 + \ln x}$ si annulla per $x = x_0$

$\int f(x)\,dx$ $\qquad f(x) = \frac{1}{g(x)}$

$f(x) = \frac{1}{g(x)} \sim ?$



$g(x_0) = 0$

$g(x) = \overset{=0}{\cancel{g(x_0)}} + \cancel{g'(x_0)} \cdot (x - x_0) + o(x - x_0)$

$g'(x) = 2x + \frac{1}{x}$

$g'(x_0) = 2x_0 + \frac{1}{x_0} > 0 \qquad x_0 > 0$

$g(x) \sim g'(x_0) \cdot (x - x_0)$ per $x \to x_0$

$f(x) \sim \frac{1}{g'(x_0) \cdot (x - x_0)}$ per $x \to x_0$

$$\int_{x_0}^{1} f(x)\,dx \quad \text{ha lo stesso carattere}$$

$$\text{di} \quad \int_{x_0}^{1} \frac{1}{g'(x_0)\cdot(x-x_0)}\,dx = +\infty$$

$$\int_0^1 \frac{1}{x^p} < +\infty \iff p<1$$

$$\rightarrow \int_{x_0}^{b} \frac{1}{(x-x_0)^p}\,dx < +\infty \iff p<1$$

$$\int_{x_0}^{1} \frac{1}{g'(x_0)\cdot(x-x_0)}\,dx$$

$$= \frac{1}{g'(x_0)} \int_{x_0}^{1} \frac{1}{x-x_0}\,dx$$

$$= \frac{1}{g'(x_0)} \Big[\ln(x-x_0)\Big]_{x_0}^{1} = \overset{>0}{\frac{1}{g'(x_0)}} \Big[\ln(1-x_0) - (-\infty)\Big]$$

$$= +\infty.$$

$$\int_{\varepsilon}^{x_0} f(x)\,dx \qquad f(x) \sim \frac{1}{g'(x_0)}\,\frac{1}{(x-x_0)}$$

$$\int_{\varepsilon}^{x_0} \underbrace{\frac{1}{g'(x_0)}}_{+}\,\underbrace{\frac{1}{(x-x_0)}}_{-}\,dx = -\infty$$

$$\int_0^{+\infty} \frac{1}{x^2+\ln x}\,dx \quad \text{non esiste}$$

D

Es $$\int_2^{+\infty} \frac{1}{x\ln x}\,dx = +\infty \qquad \frac{1}{x^p} << \frac{1}{x\ln x} << \frac{1}{x}$$

$p>1$ $\quad \int_2^{+\infty} \frac{1}{x^p} < +\infty \qquad \int_2^{+\infty} \frac{1}{x} = +\infty$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\,d\ln x$$

$$y=\ln x \quad = \int \frac{1}{y}\,dy = \ln y = \ln\ln x. \longrightarrow +\infty$$

per $x \to +\infty$.

Es $$\int_2^{+\infty} \frac{1}{x^q \ln^p x}\,dx \quad \text{per quali } p \text{ e } q \text{ converge?}$$

$$\int_0^{1/2} \frac{1}{x^q \ln^p x}\,dx \quad \text{per quali } p \text{ e } q \text{ converge?}$$

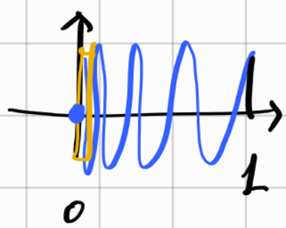
FUNZIONI A SEGNO VARIABILE

Es $$\int_0^{+\infty} \sin x\,dx = \left[-\cos x\right]_0^{+\infty} \quad \text{non esiste} \Leftarrow$$

$$= \lim_{\beta\to+\infty} (-\cos\beta) - (-\cos 0)$$

↑ non esiste

Es $$\int_0^1 \sin\frac{1}{x}\,dx \; > ?$$



$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon} \sin \frac{1}{x} \, dx$$

$$\underset{\downarrow \atop 0}{-\varepsilon} = \int_0^{\varepsilon} -1 \le \int_0^{\varepsilon} \sin \frac{1}{x} \, dx \le \int_0^{\varepsilon} 1 = \underset{\downarrow \atop 0}{\varepsilon}$$