ANALISI MATEMATICA B
LEZIONE 43 — 31.1.2021

$f: A \to B$ funzioni

$\supseteq$

C^0 funzioni continue

$\supseteq$

funzioni derivabile

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

$\supseteq$

C^1 funzioni con derivata continua

$\supseteq$

funzioni derivabile 2 volte

$\supseteq$

C^2 funzioni con derivata seconda continua

$\supseteq$

$\vdots$

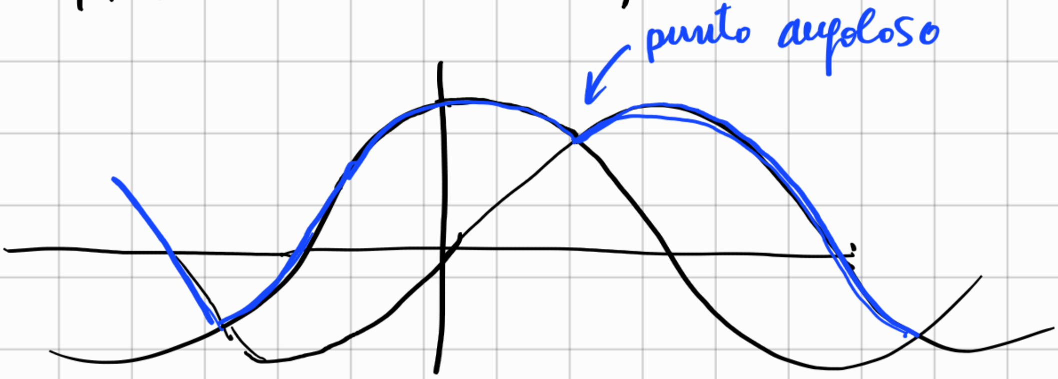
$\supseteq$

C^∞ funzioni derivabili ∞ volte

$\supseteq$

C^ω funzioni analitiche $f(x) = \sum_{k=0}^{+\infty} a_k x^k$

$f(x) = \max\{\sin x, \cos x\}$



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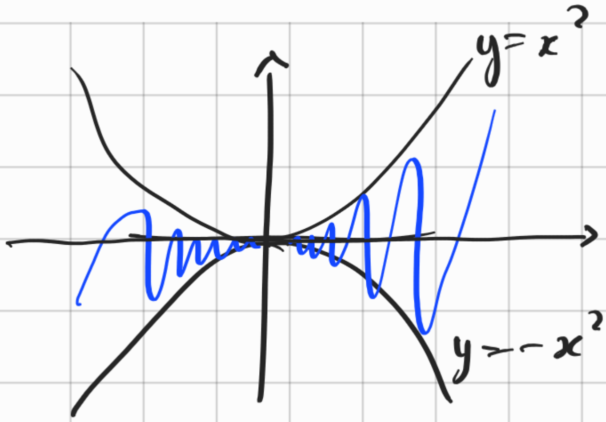


è derivabile:

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + \cos \frac{1}{x} \\ 0 & \text{se } x = 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{per } x > 0 \\ 0 & \text{per } x \leq 0 \end{cases}$$

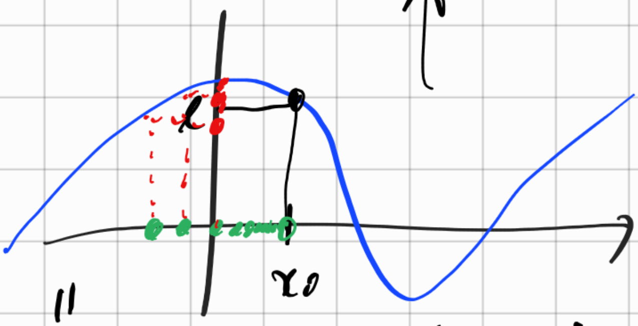
Teorema (ponte di collegamento tra limiti di funzione e limiti di successione)

$f : A \subseteq \mathbb{R}$, $A \subseteq \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ pto di accumulazione di A

$l \in \overline{\mathbb{R}}$

$\lim_{x \to x_0} f(x) = l \iff \forall a_n,\ a_n \neq x_0,\ a_n \in A:$

$a_n \to x_0 \Rightarrow \lim_{n \to +\infty} f(a_n) = l$



dim "$\Rightarrow$" è un cambio di variabile

$x = a_n$ $\quad$ $n \to +\infty \Rightarrow x = a_n \to x_0$

"$\Leftarrow$" per assurdo $\neg \lim_{x \to x_0} f(x) = \ell$

$$\neg \forall U \in \mathcal{U}_\ell : \exists V \in \mathcal{U}_{x_0} : \forall x \in (V \setminus \{x_0\}) \cap A : f(x) \in V.$$

$$\exists U \in \mathcal{U}_\ell : \underline{\forall V \in \mathcal{U}_{x_0}} : \exists x \in (V \setminus \{x_0\}) \cap A : f(x) \notin U$$

(AC)

$$\exists U \in \mathcal{U}_\ell : \forall n \in \mathbb{N} \ \exists a_n \in (V_n \setminus \{x_0\}) \cap A : f(a_n) \notin U$$

Ma $a_n \to x_0$, $a_n \neq x_0$, $a_n \in A$ $\overset{Hyp}{\Longrightarrow}$ $f(a_n) \to \ell$ IMPOSSIBILE!

V_n intorni di x_0

$$V_n = \begin{cases} \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right) & \text{se } x_0 \in \mathbb{R} \\ (n, +\infty) & \text{se } x_0 = +\infty \\ (-\infty, -n) & \text{se } x_0 = -\infty \end{cases}$$

$\square$

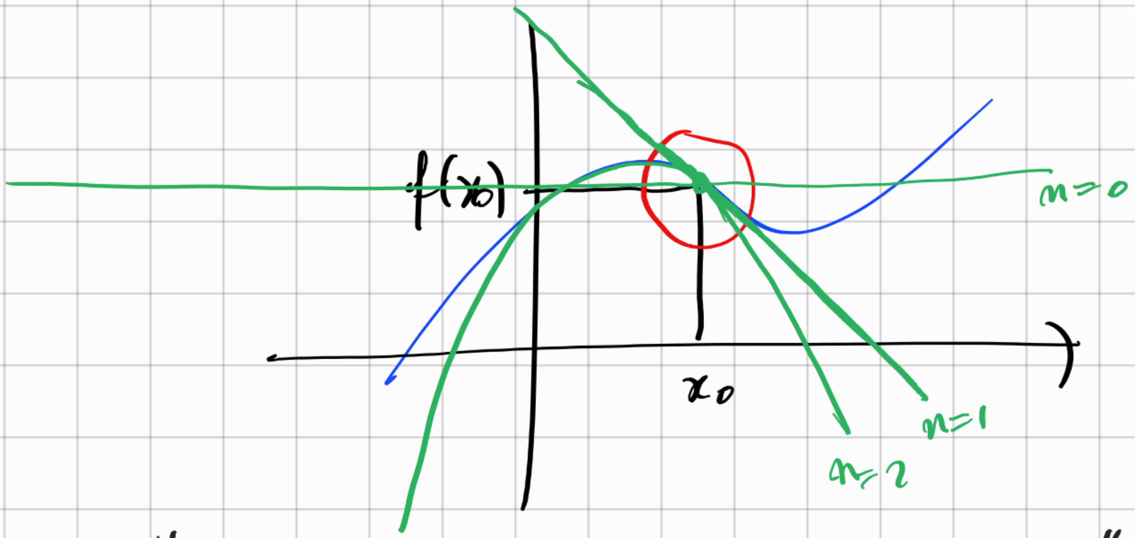
ES $\lim_{x \to +\infty} \sin x$ non esiste

$$\boxed{\lim_{n \to +\infty} \sin(\pi n) = 0}$$

$a_n = \pi n \to +\infty$

$$\lim_{n \to +\infty} \cos(\pi n) = \lim_{n \to +\infty} (-1)^n \text{ non esiste!}$$

POLINOMI di TAYLOR



$f(x)$ "derivabile n volte nel punto x_0"

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k$$

$$= f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots$$

$$\dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n .$$

è un polinomio di grado $\deg P_n \le n$.

si chiama polinomio di Taylor di ordine n

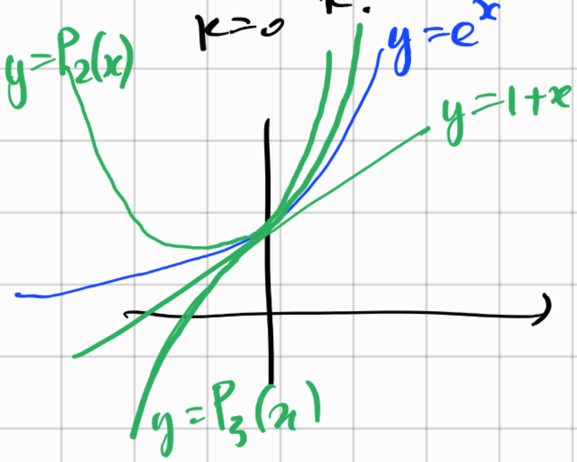
che approssima la funzione $f(x)$ nel punto x_0. (centrato in x_0)

ES $f(x) = e^x$, $x_0 = 0$, $n = 5$

$$f^{(k)}(x) = e^x, \quad f^{(k)}(x_0) = 1$$

$$P_5(x) = \sum_{k=0}^{5} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= \sum_{k=0}^{5} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$



ES $f(x) = \sin x$, $x_0 = 0$, $n = 6$

$$P_6(x) = 0 + x + 0 - \frac{x^3}{6} + 0 + \frac{x^5}{120}$$

$$= \underbrace{x - \frac{x^3}{6} + \frac{x^5}{120}}$$

$P_6 = P_5$.

ES $f(x) = \ln x$, $x_0 = 1$, $n = 3$

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{2}{x^3}$$

$$P_3(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$= x-1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$= x-1 - \frac{1}{2}(x^2 - 2x + 1) + \frac{1}{3}(x^3 - 3x^2 + 3x - 1)$$

$$= -1 - \frac{1}{2} - \frac{1}{3} + 3x - \frac{x^2}{2} - x^2 + \frac{x^3}{3} = -\frac{11}{6} + 3x - \frac{3}{2}x^2 + \frac{x^3}{3}$$

<u>Teorema</u> P_n è l'unico polinomio di grado $\leq n$ che ha le stesse derivate di f

$$\left\{ f^{(k)}(x_0) = P_n^{(k)}(x_0) \quad k = 0, 1, \dots, n. \right.$$

nel punto x_0, fino all'ordine $\underbrace{k \leq n}$.

<u>dim</u>

$$P(x) = \sum_{k=0}^{n} a_k \cdot (x-x_0)^k \qquad a_k \in \mathbb{R}.$$

$$P'(x) = \sum_{k=1}^{n} k \cdot a_k \cdot (x-x_0)^{k-1} \qquad (x-x_0)^0 = 1.$$

$$P''(x) = \sum_{k=2}^{n} k \cdot (k-1) a_k (x-x_0)^{k-2}$$

$$\vdots$$

$m \le n$ $\quad \boxed{P^{(m)}(x)} = \sum_{k=m}^{n} \overbrace{\underbrace{k\cdot(k-1)\cdots(k-m+1)}}^{m} a_k \underbrace{(x-x_0)}^{k-m}$

$\vdots$

$$P^{(n)}(x) = n! \cdot a_n$$

Es $\quad P(x) = 1+2x+x^3-4x^5$

$$P'(x) = 2+3x^2-5\cdot4\cdot x^4$$

$$P''(x) = 2\cdot3x-5\cdot4\cdot4\cdot x^3$$

$$P'''(x) = 2\cdot3-\underbrace{5\cdot4\cdot3}\cdot4x^2$$

$$P^{IV}(x) = -5\cdot4\cdot3\cdot2\cdot4x$$

$$P^{V}(x) = \underbrace{5\cdot4\cdot3\cdot2\cdot1}_{5!}\cdot\underbrace{(-4)}_{= a_5}$$

$$P^{(m)}(x_0) = m!\cdot a_m$$

se $\quad P(x) = \sum_{k=0}^{n} a_k(x-x_0)^k.$

$$P^{(m)}(x_0) = f^{(m)}(x_0)$$

$\Uparrow$

$$\Downarrow$$

$$m!\, a_m = f^{(m)}(x_0)$$

$$\Updownarrow$$

$$a_m = \boxed{\frac{f^{(m)}(x_0)}{m!}} \qquad D$$

In che senso $f(x) \simeq P_n(x)$?

Resto di Taylor

$$f(x) - P_n(x) = \underbrace{R_n(x)}$$

Teorema (formula di Taylor con resto di Peano)

f derivabile $(n-1)$ volte in un intorno di x_0 e $f^{(n-1)}$ derivabile in x_0. P polinomio di Taylor per f centrato in x_0 di ordine n.

$$\lim_{x\to x_0} \frac{f(x)-P(x)}{(x-x_0)^n} = 0 \qquad \left[\ |R_n(x)| << (x-x_0)^n\ \right]$$

dim

$$R(x) = f(x) - P(x) \qquad Q(x) = (x-x_0)^n$$

$$Q'(x) = n(x-x_0)^{n-1}$$

$$R^{(k)}(x_0) = 0 \quad \text{per } k = 0, 1, \dots, n.$$

Cauchy

$$\frac{R(x)}{(x-x_0)^n} = \frac{R(x)-R(x_0)}{(x-x_0)^n-(x_0-x_0)^n} = \boxed{\frac{R'(x_1)}{n(x_1-x_0)^{n-1}}} = (*)$$

$x > x_0$ ← ipotesi di lavoro.

$x_1 \in (x_0, x) \cup (x, x_0)$

↑ se $x < x_0$

$$R'(x) = f'(x) - P'(x)$$

Cauchy $\quad x_2 \in (x_0, x_1)$

$$(*) = \frac{R'(x_1) - R'(x_0)}{n(x_1-x_0)^{n-1} - n(x_0-x_0)^{n-1}} = \frac{R''(x_2)}{\underbrace{n(n-1)}(x_2-x_0)^{n-2}}$$

$$= \cdots\cdots = \frac{R^{(n-1)}(x_{n-1})}{\underbrace{n(n-1)\cdots\cdots 2}_{n-1}\,(x_{n-1}-x_0)} =$$

$$= \frac{f^{(n-1)}(x_{n-1}) - P^{(n-1)}(x_{n-1})}{n!\,(x_{n-1}-x_0)},$$

$$= \frac{f^{(n-1)}(x_{n-1}) - \left(f^{(n-1)}(x_0) + f^{(n)}(x_0)(x_{n-1}-x_0)\right)}{n!\,(x_{n-1}-x_0)}$$

$$= \frac{1}{n!}\left[\boxed{\frac{f^{(n-1)}(x_{n-1}) - f^{(n-1)}(x_0)}{(x_{n-1}-x_0)}} - f^{(n)}(x_0)\right]$$

$\to f^{(n)}(x_0)$

$$x \to x_0,\ x_1 \to x_0,\ x_2 \to x_0, \ldots, x_{n-1} \to x_0$$

$$\longrightarrow 0$$

$\square$