ANALISI MATEMATICA B
LEZIONE 43 - 17.1.2022

$$\lim_{n\to+\infty} \sum_{k=1}^{n^2} \frac{1}{(n^3+k)^\alpha} \qquad \text{oppure } \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{(n^3+k^2)^\alpha}$$

$$\frac{1}{(n^3+n^2)^\alpha} \le \left[\frac{1}{(n^3+k)^\alpha}\right] \le \frac{1}{(n^3)^\alpha} \sim \frac{1}{n^{3\alpha}}$$

$$\underbrace{\frac{n^2}{(n^3+n^2)^\alpha}}_{\sim\, n^{2-3\alpha}} = \sum_{k=1}^{n^2} \frac{1}{(n^3+n^2)^\alpha} \le \sum_{k=1}^{n^2} \frac{1}{(n^3+k)^\alpha} \le \sum_{k=1}^{n^2} \frac{1}{(n^3)^\alpha} = \frac{n^2}{n^{3\alpha}} = n^{2-3\alpha}$$

$$n^{2-3\alpha} \xrightarrow[n\to+\infty]{} \begin{cases} +\infty & \text{se } \alpha < \frac{2}{3} \\ 1 & \text{se } \alpha = \frac{2}{3} \\ 0 & \text{se } \alpha > \frac{2}{3} \end{cases} \qquad 2-3\alpha > 0$$

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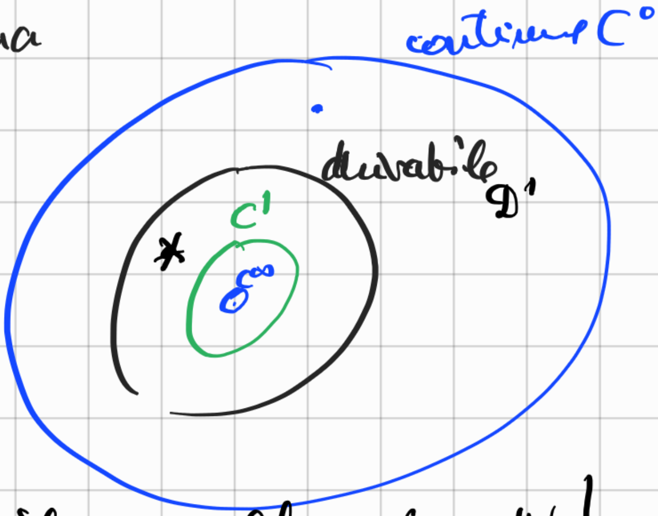
f derivabile $\Rightarrow$ f continua

$\not\Downarrow$

f' continua?



Se f è derivabile e f' continua diremo che f è C^1

$$C^\infty = \bigcap_{k=1}^{\infty} C^k = \{ f : f \text{ è derivabile } n \text{ volte } \forall n \in \mathbb{N} \}$$

Esempio (*)

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

Se $x \neq 0$ $\quad f'(x) = \left(x^2 \sin \frac{1}{x} \right)' = 2x \sin \frac{1}{x} + x^2 \left(\cos \frac{1}{x} \right) \cdot \left(-\frac{1}{x^2} \right)$

$$= 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

$\lim_{x \to 0} f'(x)$ non esiste.

Se $x = 0$

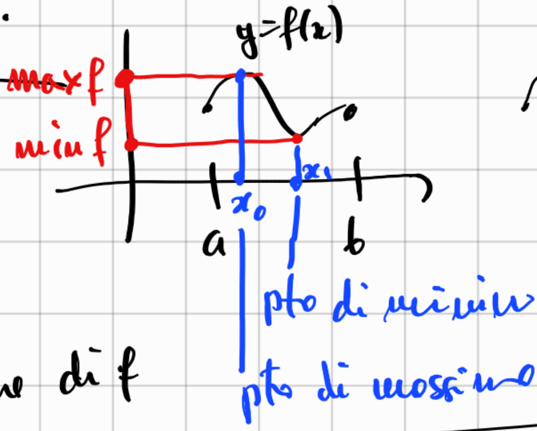
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\cancel{2}} \sin \frac{1}{h} - 0}{\cancel{h}} = 0$$

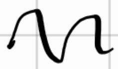
$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

f è derivabile ma f' non è continua. $\square$

Teorema (Weierstrass) $f:[a,b]\to\mathbb{R}$ continua. $(a\le b)$

f ha massimo e minimo.





$$\max_{[a,b]} f = \max\{f(x): x\in[a,b]\}$$
$$= \max \underbrace{f([a,b])}_{\text{immagine di } f}$$

$[a,b]$ è chiuso e limitato

$[a,+\infty)$ "è chiuso in $\mathbb{R}$" ma non è limitato

$= \{x\in\mathbb{R} : x\geq a\}$

$[a,b)$ è limitato ma non è chiuso

Es 1 $\quad f:[0,1)\to\mathbb{R} \qquad f(x)=\dfrac{1}{x-1}$

f è continua

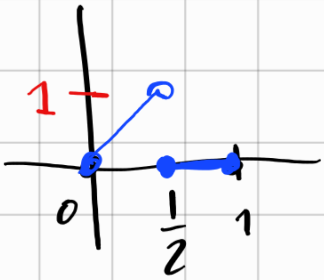
ma non ha minimo

Es 2 $\quad f:[0,+\infty)\to\mathbb{R} \qquad f(x)=x$

f è continua ma non ha massimo

Es 3 $\quad f:[0,1]\to\mathbb{R}$

$$f(x)=\begin{cases} x & \text{se } x<\frac{1}{2} \\ 0 & \text{se } x\geq\frac{1}{2}\end{cases}$$



f non ha massimo

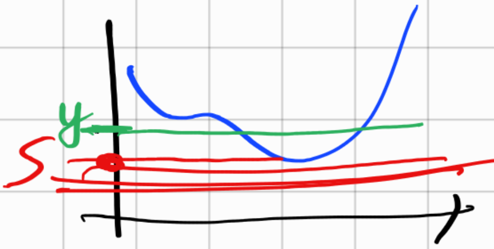
dim $\quad f:[a,b]\to\mathbb{R}$

B-W: $x_k\in\mathbb{R}$, x_k limitata $(a\le x_k\le b)$
$\exists\ x_{k_j}$ sottosuccessione convergente.

Dimostriamo che f ha minimo.

$$s=\inf f([a,b]) \qquad s\in[-\infty,+\infty)$$

- considero una "successione minimizzante" cioè $x_k\in[a,b]$ t.c. $f(x_k)\to s$ per $k\to+\infty$



$$s=\inf f([a,b])$$
$$\Updownarrow$$
(i) $s\le f(x)\ \forall x\in[a,b]$

(ii) se $y>s$ $\exists\, x\in[a,b]:\ \underset{\overset{\vee\!|}{s}\ (i)}{f(x)}<y$.

$$\forall y>s\ \exists\, x\in[a,b]:\quad s\le f(x)<y$$

Prendo $y_k\to s$ $\quad \exists\, x_k\in[a,b]\quad s\le f(x_k)<\underset{\downarrow\, s}{y_k}$

(es. se $s>-\infty \quad y_k=s+\frac{1}{k}$
$\quad$ se $s=-\infty \quad y_k=-k$)

$\exists\, x_k\in[a,b]: f(x_k)\to s \implies f(x_{k_j})\to s =$

B-W: $\exists\, k_j \qquad x_{k_j}\to x\in[a,b]\subseteq\mathbb{R}$

f continua: $\quad f(x_{k_j})\to f(x) =$

$$\Downarrow$$

$$\lim_{t\to x} f(t) = f(x) \Rightarrow x_k \to x \quad \lim_{k\to+\infty} f(x_k) = f(x)$$

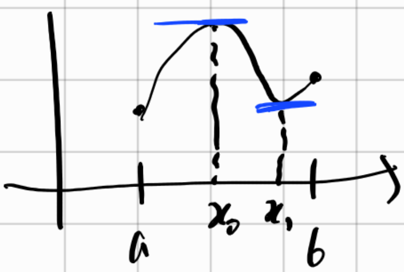
$$f(x) = \inf f([a,b])$$
$$= \min f([a,b]) \qquad \square$$

Def $A \subseteq \mathbb{R}$ è chiuso (in $\mathbb{R}$) se ogni volta che prendo $x_k \in A$ t.c. $x_k \to x \in \mathbb{R}$.

Allora $x \in A$.

Def $A \subseteq \mathbb{R}$ si dice essere "compatto" se è chiuso e limitato.
(da ogni successione $x_k \in A$ posso estrarre una sottosuccessione convergente a un punto dell'insieme).

Teo (Fermat) $f:(a,b)\to\mathbb{R}$. Supponiamo che abbia massimo o minimo in un punto $x_0 \in (a,b)$. Se f è derivabile in x_0 allora $f'(x_0) = 0$.

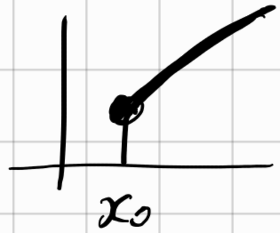


dim Sia x_0 un punto di massimo: $f(x) \le f(x_0)$ $\forall x \in (a,b)$

$$f'(x_0) \xleftarrow[x\to x_0]{} \frac{f(x)-f(x_0)}{x-x_0} = \text{"}\frac{-}{\pm}\text{"} = \begin{cases} \le 0 & \text{se } x > x_0 \\ \ge 0 & \text{se } x < x_0 \end{cases}$$

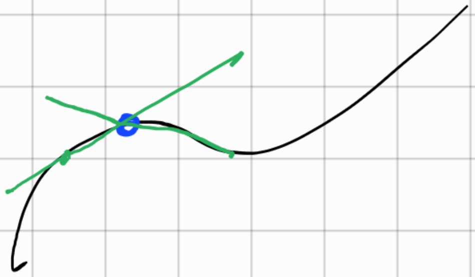
$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$$

$$f'(x_0) = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

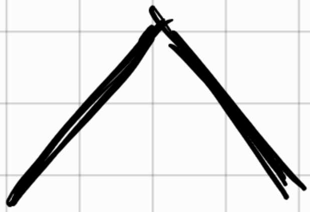


$$\Rightarrow f'(x_0) = 0. \quad \square$$

$x_0 \in (a,b)$

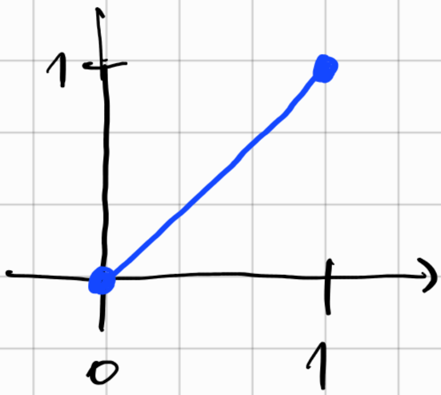


Es



Es: $f: [0,1] \to \mathbb{R}$ $\quad f(x) = x$

ha massimo per $x = 1$
ma $f'(1) = 1$.

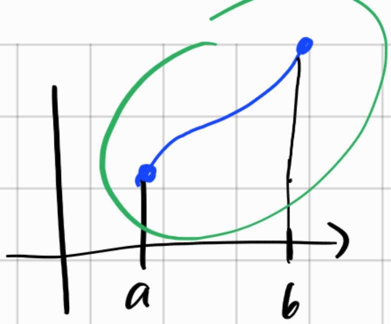


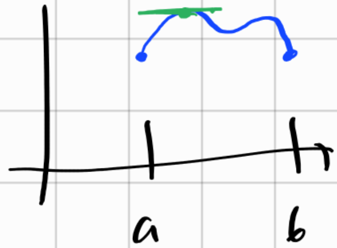
Es $\quad f(x) = ax^2 + bx + c$ ha massimo o minimo nel vertice.

$$f'(x) = 2ax + b = 0 \qquad x = -\frac{b}{2a}.$$

Teorema (Rolle) Sia $f: [a,b] \to \mathbb{R}$ continua, derivabile in (a,b). Se $f(a) = f(b)$ allora

$$\exists\, x_0 \in (a,b) \text{ t.c. } f'(x_0) = 0.$$





dim Per Weierstraß f ha massimo e minimo su $[a,b]$. Sia $x_0 \in [a,b]$ il pto di massimo e $x_1 \in [a,b]$ il pto di minimo.

Se $x_0 \in (a,b) \overset{\text{Fermat}}{\Longrightarrow} f'(x_0) = 0$ ✓

Altrimenti $x_0 \in \{a,b\}$.

Se $x_1 \in (a,b) \Rightarrow f'(x_1) = 0$ ✓

Altrimenti $x_1 \in \{a,b\}$

$$\Rightarrow \underbrace{\underset{\max f}{f(x_0)} = \underset{\min f}{f(x_1)}}_{f \text{ costante}} \qquad (f(a) = f(b))$$

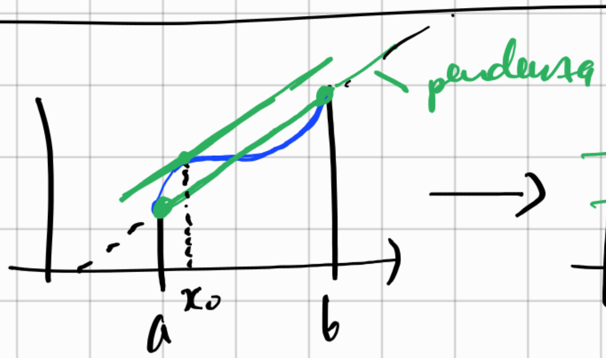
$f(x) = f(a) = f(b)$

$\Rightarrow f'(x) = 0 \quad \forall x \in (a,b)$. □

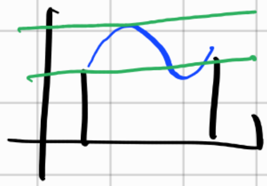
Teorema di Lagrange

$f : [a,b] \to \mathbb{R}$, f continua in $[a,b]$, f derivabile in (a,b).

$$\exists\, x_0 \in (a,b) : \qquad f'(x_0) = \frac{f(b) - f(a)}{b - a} = m$$



$\longrightarrow$



dimm $g(x) = f(x) - mx \qquad m = \dfrac{f(b) - f(a)}{b-a}$

$$g(b) = f(b) - \frac{f(b)-f(a)}{b-a}\, b = \frac{(\cancel{b}-a)f(b) - \cancel{bf(b)} + bf(a)}{b-a}$$

$$g(a) = f(a) - \frac{f(b)-f(a)}{b-a}\, a = \frac{(b-\cancel{a})f(a) - af(b) + \cancel{af(a)}}{b-a} =$$

$$g(a) = g(b) \overset{\text{Rolle}}{\Longrightarrow} \exists\, x_0 : g'(x_0) = 0$$

$$\Downarrow$$

$$g'(x) = f'(x) - m \qquad f'(x_0) - m$$

$$f'(x_0) = m \quad \square$$