ANALISI MATEMATICA B
LEZIONE 42 — 14.1.2022

$$f'(x_0) = \lim_{x \to x_0} \boxed{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$h = x - x_0$

<u>Es</u> $f(x) = \frac{1}{x}$ $\qquad f'(x) = -\frac{1}{x^2}$

$h(x) = g(f(x)) \qquad h'(x) = g'(f(x)) \cdot f'(x)$

<u>Notazione</u> $f(x)$

se f ha più variabili

$$f'(x) = \frac{df}{dx}(x) = Df(x) \left[= \frac{\partial f}{\partial x}(x) \right]$$

$$f'(x) = (f(x))'$$

<u>Es</u> $(g \circ f)' = (g' \circ f) \cdot f'$

$$D(g \circ f) = (Dg) \circ f \cdot Df$$

Teorema (derivata della funzione inversa)

$f: A \to B \subseteq \mathbb{R}$ bigettiva $A \subseteq \mathbb{R}$

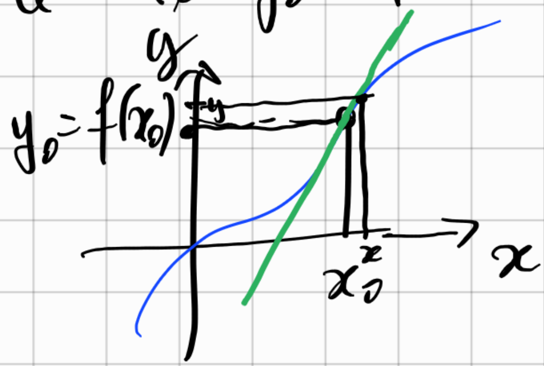
f derivabile in $x_0 \in A$, $f'(x_0) \neq 0$.

f^{-1} continua in $y_0 = f(x_0)$

Allora f^{-1} è derivabile in y_0

e $$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

dim f^{-1} derivabile in $y_0 = f(x_0)$?



$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} =$$

$y = f(x)$
$x = f^{-1}(y)$

$$= \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}$$

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \stackrel{!}{=} \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = (*)$$

se $y \to y_0$ allora $x \to x_0$?

$$x = f^{-1}(y)$$
$$x_0 = f^{-1}(y_0)$$

$\Leftarrow$ $\boxed{f^{-1}(y) \to f^{-1}(y_0) \text{ per } y \to y_0}$

$$* = \frac{1}{f'(x_0)} \quad \text{se } f'(x_0) \neq 0 \qquad \square$$

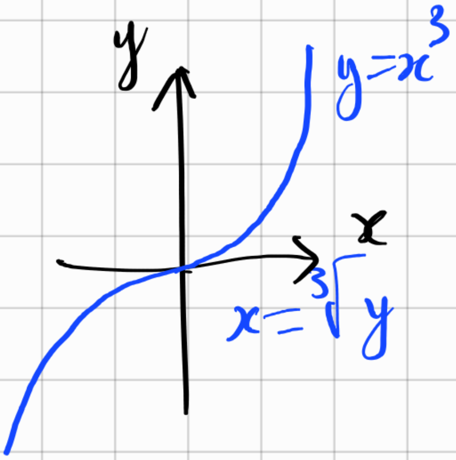
<u>Nota</u> se $f'(x_0) = 0$ f^{-1} non è derivabile in y_0

<u>Es</u> $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2 \cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}} - \cancel{x^3}}{\cancel{h}}$$

$$= \lim_{h \to 0} \left(3x^2 + \underbrace{\cancel{3xh}}_{0} + \underbrace{\cancel{h^2}}_{0}\right) = 3x^2.$$

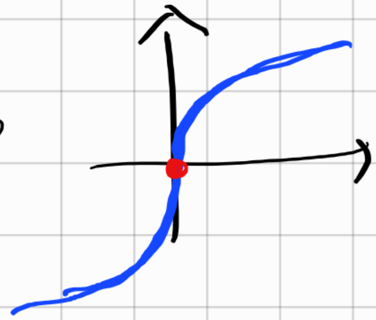


$$g(y)=f^{-1}(y)=\sqrt[3]{y}$$

$$g'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(\sqrt[3]{y})} = \frac{1}{3(\sqrt[3]{y})^2}$$

$y \neq 0$

$g(y)$ non è derivabile in $y=0$



Operazioni con le derivate:

Se f e g sono derivabili in x_0 $\quad c \in \mathbb{R}$

Allora $c \cdot f$, $f+g$, $f-g$, $f \cdot g$ e $\frac{f}{g}$ (se $g(x_0) \neq 0$) sono derivabili in x_0 e vale:

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

$$(c \cdot f)'(x_0) = c \cdot f'(x_0)$$

$f \xmapsto{D} f'$

D è un operatore lineare.

$$(f-g)'(x_0) = f'(x_0) - g'(x_0)$$

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g^2(x_0)}$$

dim

$f(x)+g(x)$

$$\frac{(f(x) \pm g(x)) - (f(x_0) \pm g(x_0))}{x-x_0} = \boxed{\frac{f(x)-f(x_0)}{x-x_0}} \xrightarrow{} f'(x_0) \pm \boxed{\frac{g(x)-g(x_0)}{x-x_0}} \xrightarrow{} g'(x_0)$$

$$\frac{c\cdot f(x) - c f(x_0)}{x-x_0} = c\cdot \boxed{\frac{f(x)-f(x_0)}{x-x_0}} \xrightarrow{} f'(x_0)$$

$f(x)\cdot g(x)$

$$\frac{f(x)\cdot g(x) - f(x_0)\cdot g(x_0)}{x-x_0} = \frac{f(x)\cdot g(x) - f(x)\cdot g(x_0) + f(x)\cdot g(x_0) - f(x_0)\cdot g(x_0)}{x-x_0}$$

$$= f(x)\cdot \frac{g(x)-g(x_0)}{x-x_0} + \frac{f(x)-f(x_0)}{x-x_0}\cdot g(x_0)$$

$x \to x_0$

$$\to f(x_0)\cdot g'(x_0) + f'(x)\cdot g(x_0)$$

$$\frac{f(x)}{g(x)} = f(x)\cdot \underbrace{h(g(x))}_{p}$$

$h(y) = \frac{1}{y}$

$h'(y) = -\frac{1}{y^2}$

$$\left(\frac{f}{g}\right)'(x_0) = f'(x)\cdot h(g(x)) + f(x)\cdot h'(g(x))\cdot g'(x)$$

$$= \frac{f'(x)}{g(x)} + f(x)\cdot\left(-\frac{1}{g^2(x)}\right)\cdot g'(x)$$

$$= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)} \qquad \square$$

Derivate delle funzioni elementari.

$$f(x) = m\cdot x + q \qquad f'(x) = m \qquad (D\cdot c = 0)$$

$$\left[D\left(m\cdot f(x) + q\right) = m\cdot f'(x) \right]$$

$n \in \mathbb{N}, n > 0$

$\rightarrow$ $f(x) = x^n \qquad f'(x) = n\cdot x^{n-1}$

per induzione:

(i) $n = 1 \qquad f(x) = x \qquad f'(x) = 1$

$n\cdot x^{n-1} = 1\cdot x^0 = 1.$

(ii) $(x^n)' = n x^{n-1}$

$$\left(x^{n+1}\right)' = \left(x\cdot x^n\right)' = 1\cdot x^n + x\cdot n\cdot x^{n-1} = (n+1)\cdot x^n \quad \checkmark$$

$\rightarrow$ $f(x) = x^{-n} = \frac{1}{x^n}$

$$f'(x) = -\frac{1}{(x^n)^2}\cdot n x^{n-1} = -n x^{(n-1)-2n} = -n\cdot x^{-n-1}$$

$\rightarrow$ $\left[f(x) = x^{\alpha} \qquad f'(x) = \alpha\cdot x^{\alpha-1} \qquad \text{DOPO} \right]$

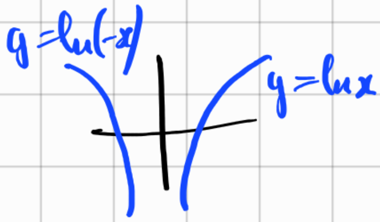
$$f(x) = e^x \qquad f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x\cdot \underbrace{\boxed{\frac{e^h - 1}{h}}}_{1}$$

$= e^x$

$f(x) = \ln x \qquad g(y) = e^y \qquad g'(y) = e^y$

$$f'(x) = \frac{1}{e^{\ln x}} = \frac{1}{x} \qquad f'(x) = \frac{1}{g'(f(x))}$$

$(x>0)$



$$\left[\; f(x) = \ln|x| \right.$$

$$\left. = \begin{cases} \frac{1}{x} & \text{se } x>0 \\ \frac{1}{-x}\cdot(-1) = \frac{1}{x} & \text{se } x<0 \end{cases} = \frac{1}{x} \quad (x \neq 0) \;\right]$$

$$f(x) = x^{\alpha} = e^{\alpha \cdot \ln x} \qquad (x>0)$$

$$f'(x) = e^{\alpha \cdot \ln x} \cdot \alpha \cdot \frac{1}{x} = x^{\alpha} \cdot \alpha \cdot \frac{1}{x} = \alpha \cdot x^{\alpha - 1}.$$

$$\rightarrow \left(x^{\frac{1}{n}}\right)' = \underbrace{\frac{1}{n} \cdot x^{\frac{1}{n}-1} = \frac{1}{n}\, x^{\frac{1-n}{n}}}$$

$$\boxed{f(x) = \sqrt[n]{x}}$$

$$f'(x) = \boxed{\frac{1}{n}\sqrt[n]{x^{1-n}}} \leftarrow \quad x \neq 0$$

$n=2 \qquad (\sqrt{x})' = \frac{1}{2}\sqrt{x^{-1}} = \boxed{\frac{1}{2\sqrt{x}}}$

$$f'(x) = \left(n\left(\sqrt[n]{x}\right)^{n-1}\right)^{-1} \qquad g(y) = y^n$$

$$g'(y) = ny^{n-1}$$

$$= \frac{1}{n}\,\frac{1}{\sqrt[n]{x^{n-1}}} = \frac{1}{n}\sqrt[n]{\frac{1}{x^{n-1}}}$$

$$= \frac{1}{n}\sqrt[n]{x^{1-n}} \qquad \checkmark$$

$$f(x) = \sin x$$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left[\sin x \cdot \underbrace{\frac{\cos h - 1}{h}}_{\to 0} + \cos x \cdot \underbrace{\frac{\sin h}{h}}_{\to 1} \right] = \cos x$$

$$f(x) = \cos x$$

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left[\cos x \underbrace{\frac{\cos h - 1}{h}}_{\to 0} - \sin x \underbrace{\frac{\sin h}{h}}_{\to 1} \right] = -\sin x.$$

$$f(x) = |x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \end{cases}$$

f non è derivabile in $x = 0$

$$= \frac{x}{|x|}$$

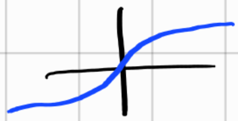
Es $$\left(\sqrt{|x|}\right)' = \frac{1}{2\sqrt{|x|}} \cdot \frac{x}{|x|} = \frac{x}{2\sqrt{|x|^3}}.$$

$$f(x) = \operatorname{tg} x = \frac{\sin x}{\cos x}$$

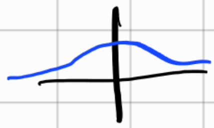
$$f'(x) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \begin{cases} \dfrac{1}{\cos^2 x} \\ 1 + \operatorname{tg}^2 x \end{cases}$$

$$f(x) = \operatorname{arctg} x$$



$$f'(x) = \frac{1}{1 + \operatorname{tg}^2(\operatorname{arctg} x)} = \frac{1}{1+x^2}$$

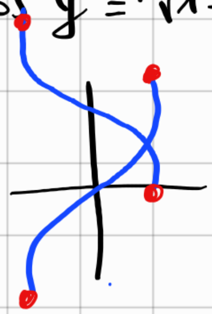


$$-1 \le x \le 1$$

$$-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$$

$$f(x) = \arcsin x$$

$$\cos y = +\sqrt{1 - \sin^2 y}$$



$$f'(x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}}$$

$$-1 < x < 1$$

$$f(x) = \arccos x$$

$$f'(x) = -\frac{1}{\sqrt{1-x^2}}$$
