ANALISI MATEMATICA B
LEZIONE 74 - 14.4.2021

Metodi risolutivi per le equazioni differenziali

eq. del I ordine:

- eq lineari: $u'(x) + a(x)u(x) = b(x)$

  metodo: moltiplico per $e^{A(x)}$ $\quad A \in \int a$.

  $$(u \cdot e^{A(x)})' = b(x) e^{A(x)}$$

- eq a variabili separabili:

  $$u'(x) = g(x) \cdot h(u(x))$$

metodo: controllo $h(u) = 0$ → soluzioni stazionarie

dividendo per $h(u(x))$:

$$\frac{u'(x)}{h(u(x))} = g(x)$$

$$\left(H(u(x))\right)' = g(x) \qquad H = \int \frac{du}{h(u)}$$

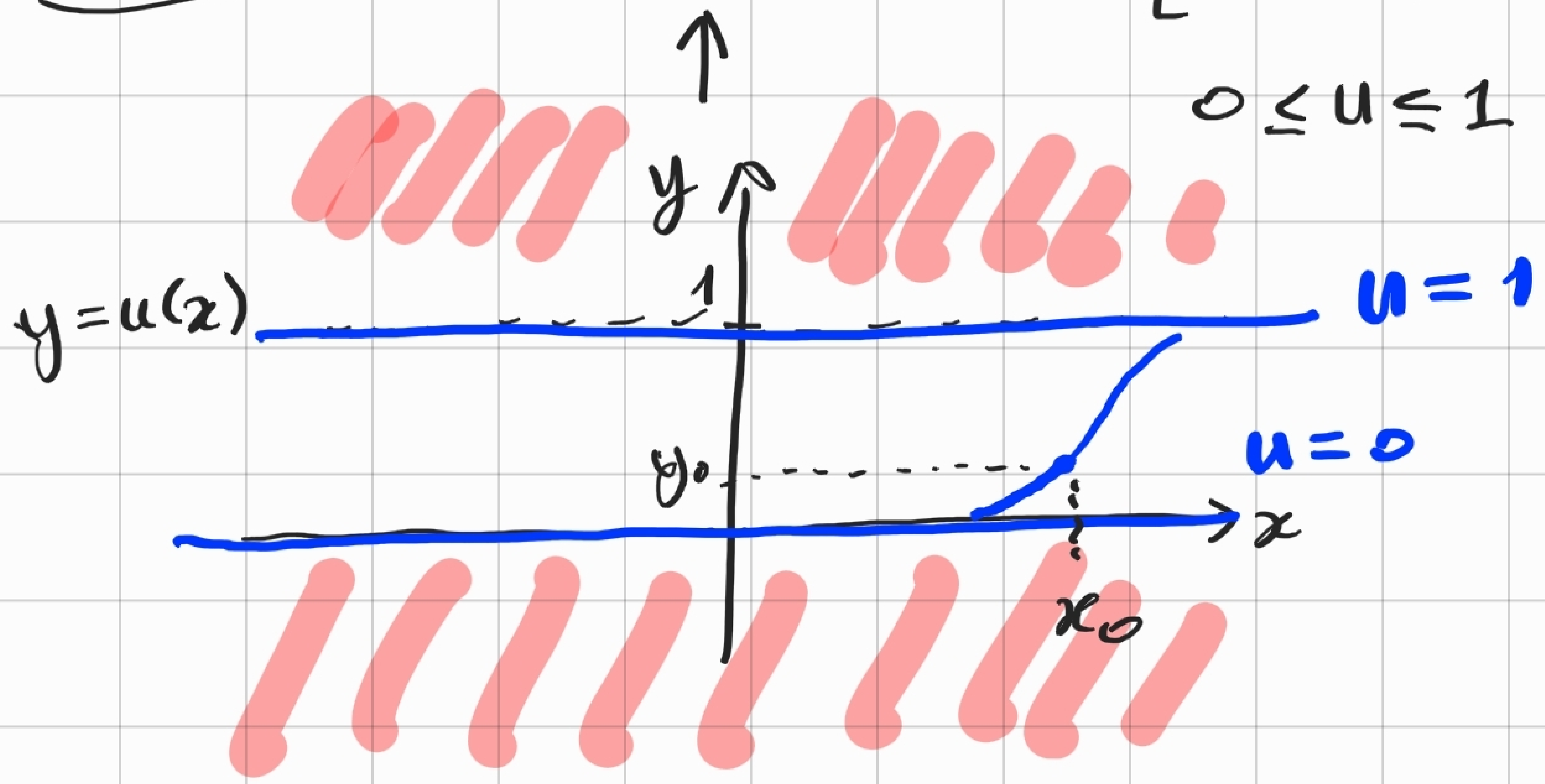
$$\int \frac{du}{h(u)} = \int g(x) + c$$

$$H(u(x)) = \int g(x) + c$$

$$u(x) = H^{-1}(\cdots\cdots)$$

Esercizio $\quad u' = \sqrt{u-u^2} \quad \left[u'(x) = \sqrt{u(x)-u^2(x)} \right]$

$0 \le u \le 1$



$u - u^2 = 0 \qquad u(1-u) = 0 \qquad u=0,\ u=1$

Se in un certo punto x_0 ho che $u(x_0) - u^2(x_0) > 0$

cioè $\quad 0 < u(x_0) < 1$

allora posso dividere ambo i membri per $\sqrt{u-u^2}$:

$$\boxed{\frac{u'(x)}{\sqrt{u(x)-u^2(x)}} = 1} \quad \leftarrow \forall x \text{ in un intorno di } x_0$$

$$\int_{x_0}^{x} \frac{u'(t)}{\sqrt{u(t)-u^2(t)}}\, dt = \int_{x_0}^{x} 1\, dt$$

$$y = u(t) \qquad dy = u'(t)\,dt$$

(*) $$\int_{u(x_0)}^{u(x)} \frac{1}{\sqrt{y-y^2}}\,dy = x - x_0$$

$$= \int \frac{1}{\sqrt{u-u^2}}\,du \qquad u^2-u = \left(u-\frac{1}{2}\right)^2 - \frac{1}{4}$$

$$v = u - \frac{1}{2} \qquad u = v + \frac{1}{2}$$

$$u-u^2 = \frac{1}{4} - v^2 \qquad dv = du$$

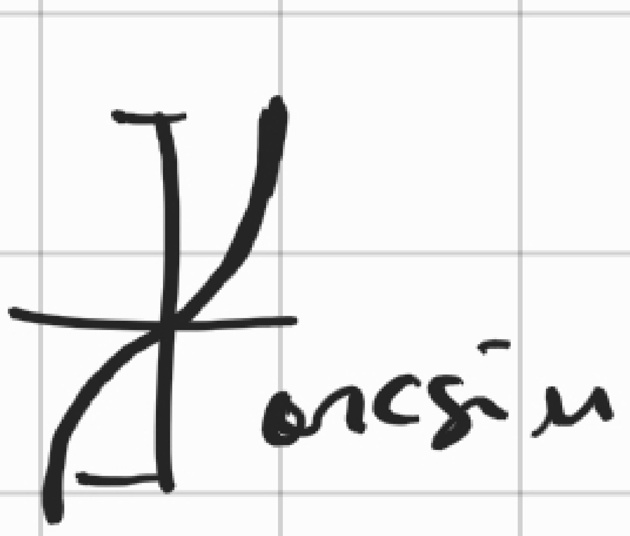
$$= \int \frac{1}{\sqrt{\frac{1}{4} - v^2}}\,dv = \int \frac{2}{\sqrt{1-(2v)^2}}\,dv \Rightarrow \arcsin(2v) = \arcsin(2u-1)$$

(*) $$\Big[\arcsin(2y-1)\Big]_{u(x_0)}^{u(x)} = x - x_0$$

$$\arcsin\big(2u(x)-1\big) - \arcsin\big(2u(x_0)-1\big) = x - x_0$$

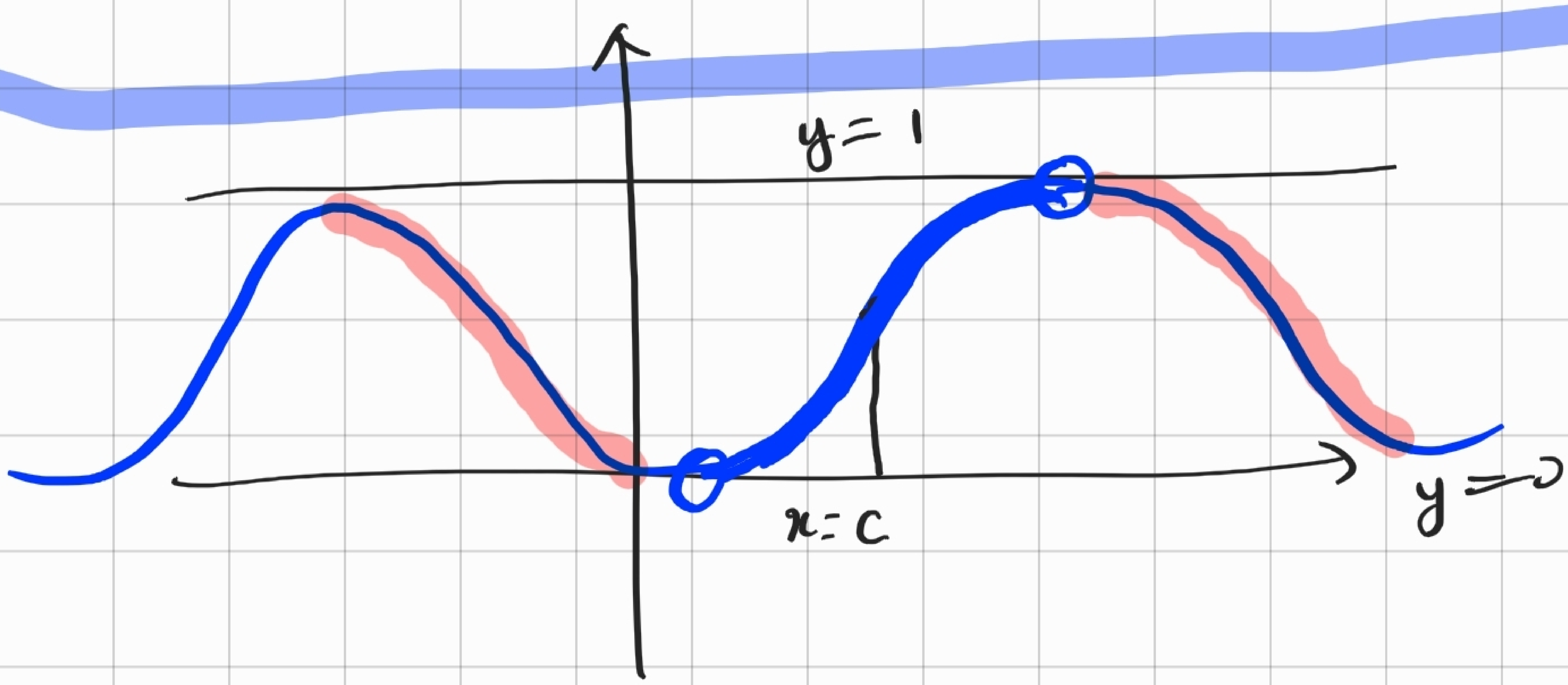
$$\arcsin\big(2u(x)-1\big) = x - c$$

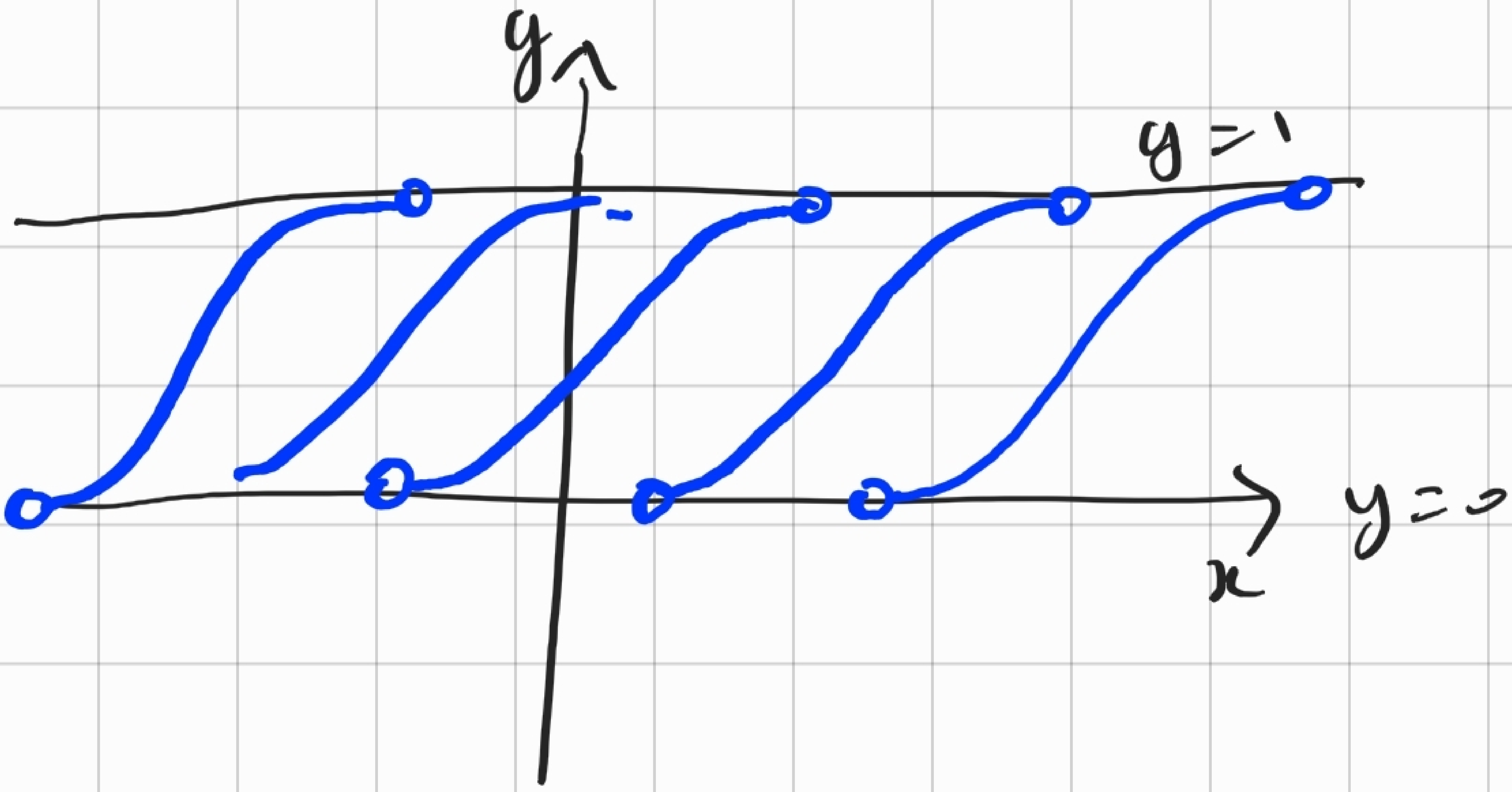
$$2u(x) - 1 = \sin(x-c)$$

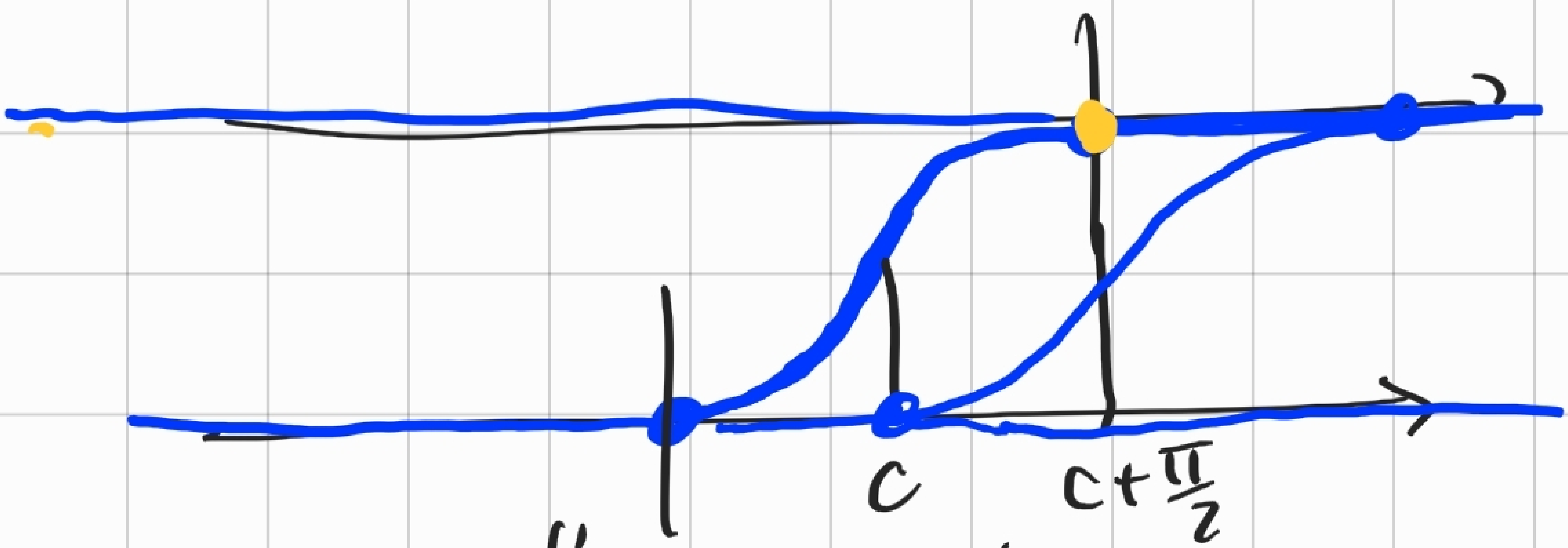


$$x - c \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\boxed{u(x) = \frac{1+\sin(x-c)}{2} \qquad x \in \left(c-\frac{\pi}{2}, c+\frac{\pi}{2}\right)}$$







La soluzione "nominale" è:

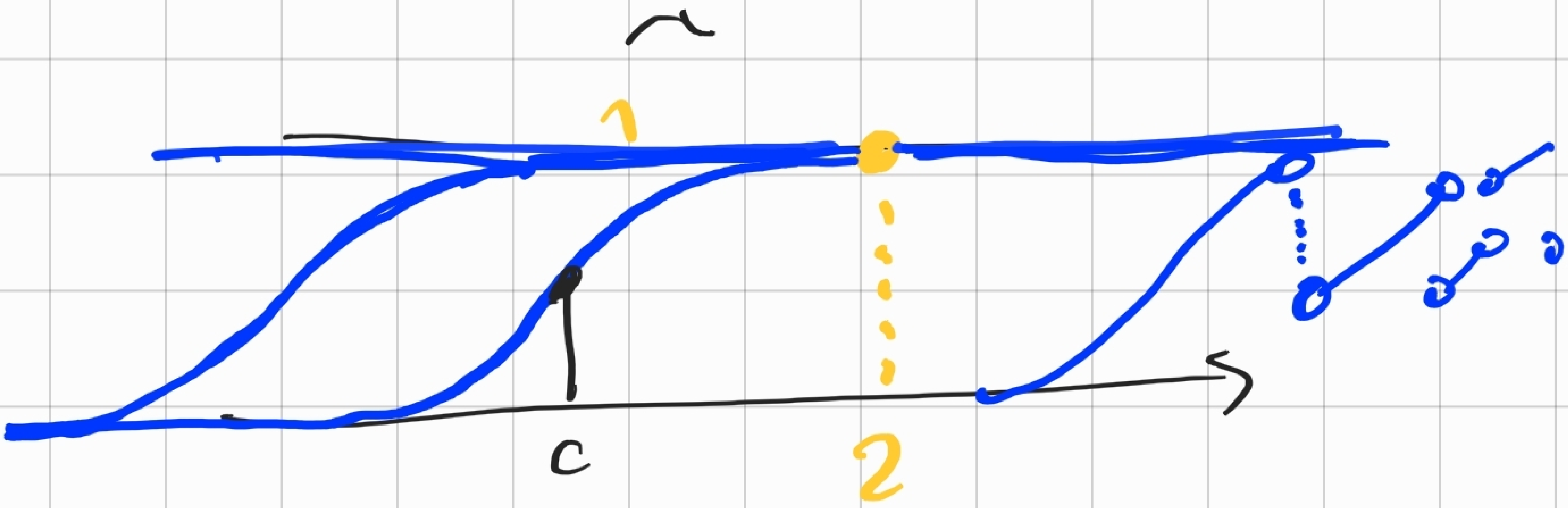
$$\longrightarrow u_c(x) = \begin{cases} 1 & \text{se } x \geq c+\frac{\pi}{2} \\ \frac{1+\sin(x-c)}{2} & \text{se } x \in \left[c-\frac{\pi}{2}, c+\frac{\pi}{2}\right] \\ 0 & \text{se } x \leq c-\frac{\pi}{2} \end{cases}$$

u è continua (anche in $c+\frac{\pi}{2}$ e $c-\frac{\pi}{2}$)

u è derivabile con derivata continua.

u è soluzione. $\square$

Non c'è unicità!



$$\begin{cases} u'(x) = \sqrt{u(x) - u^2(x)} \\ u(2) = 1 \end{cases}$$

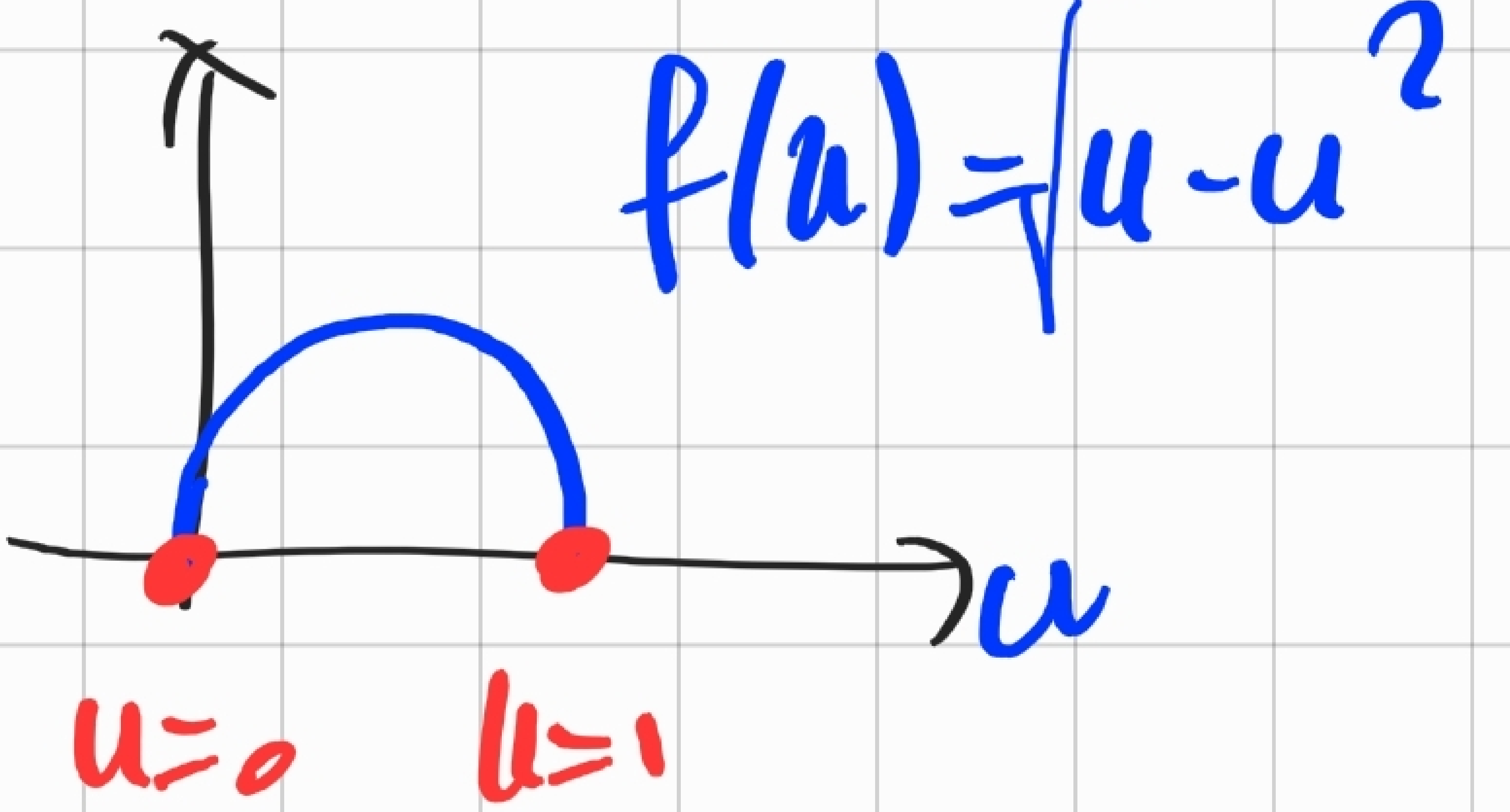
$u_1(x) \equiv 1$

$u_c(x)$ con $c \leq 2 - \frac{\pi}{2}$

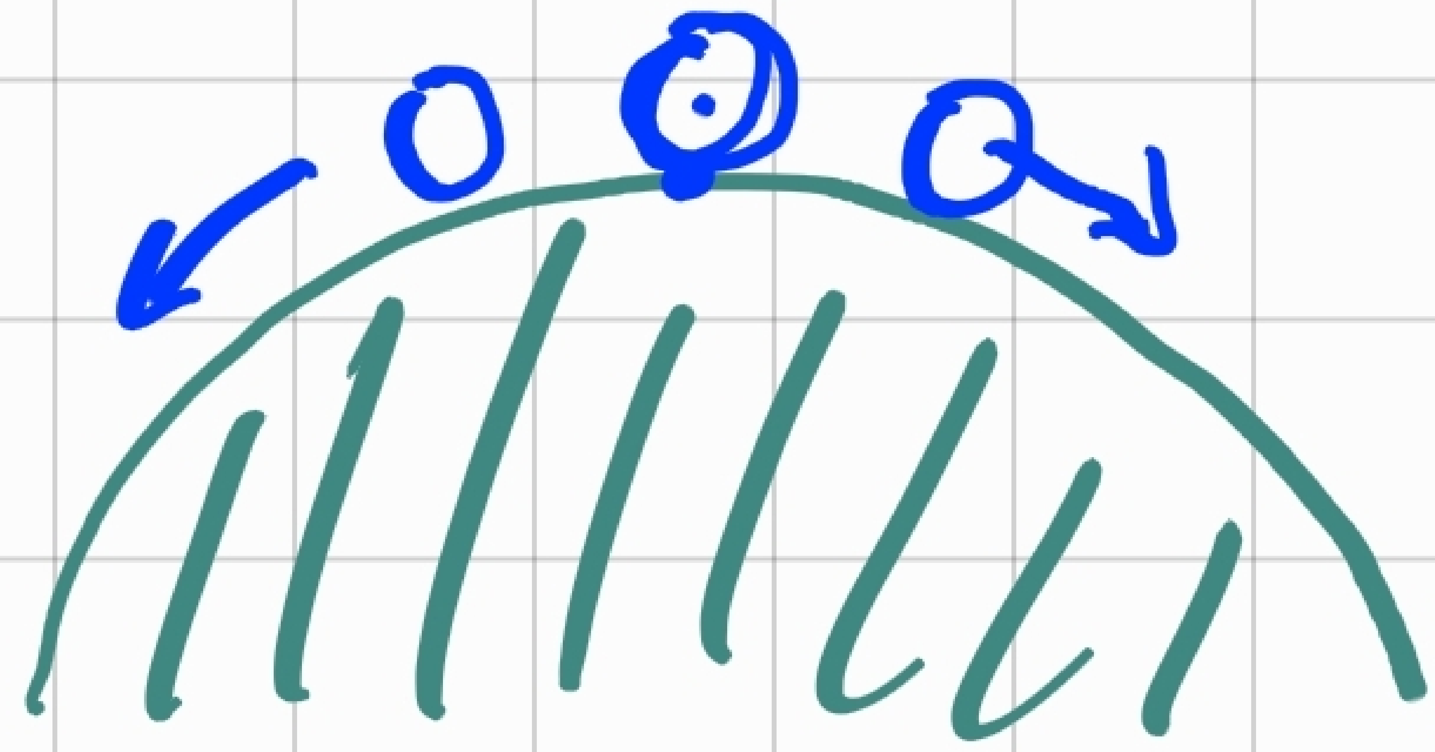
$$u'(x) = f(x, u(x))$$

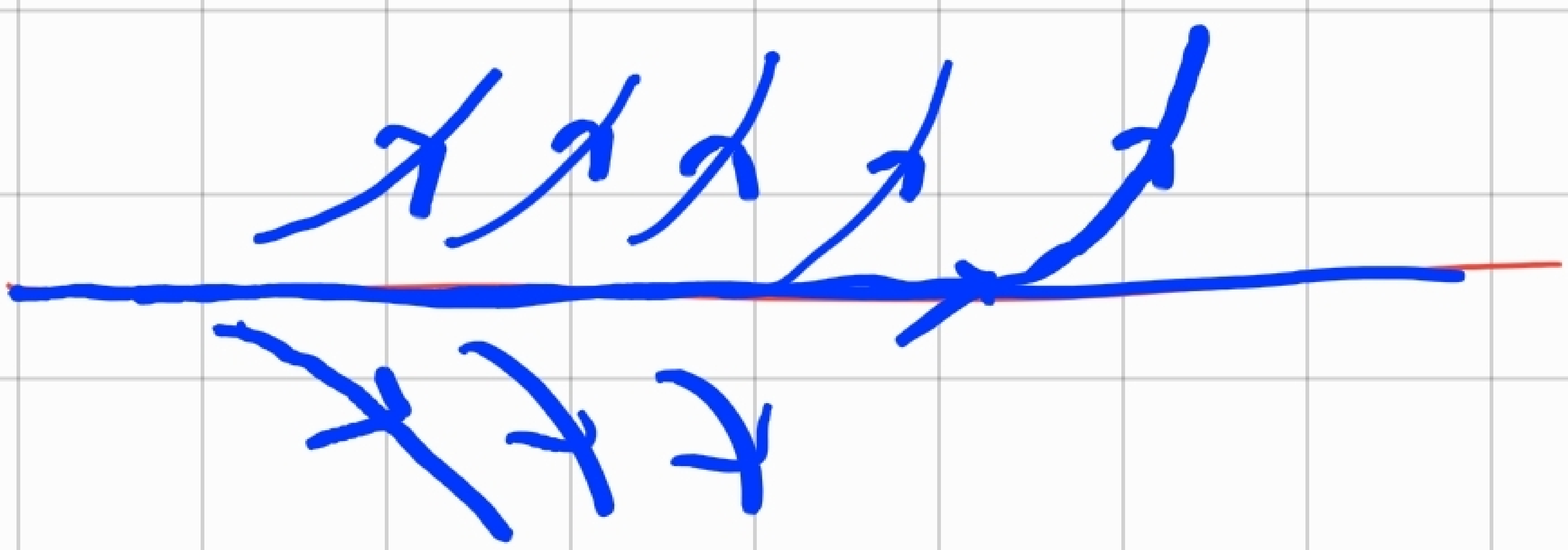
con f non regolare.

$f \in C^\infty$ solo se $0 < u < 1$



$$\boxed{u' = \sqrt[3]{u}}$$





Equazioni differenziali lineari a coefficienti costanti.

$$\underbrace{u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \dots + a_2 u''(x) + a_1 u'(x) + a_0 u(x)}_{Lu} = b(x)$$

$a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ (costanti) coefficienti

$b(x)$ termine noto.

Se $b(x) \equiv 0$ l'equazione si dice omogenea.

$$L[u] = b$$

$$L[u] = \sum_{k=0}^{n} a_k \cdot u^{(k)} \qquad (a_n = 1)$$

ES: $u'' + u = 0$ moto armonico.

$u'' + u = f(x)$ moto " forzato

Visto che $u_1 = \sin x$ e $u_2 = \cos x$

risolvono $u'' + u = 0$

Tutte le funzioni $u = A \cdot u_1 + B \cdot u_2$

sono soluzioni.

Per un teorema generale lo spazio delle soluzioni ha dimensione n = ordine della equazione $= 2$.

Quindi $u(x) = A \cdot \sin x + B \cos x$ sono tutte le soluzioni di $u'' + u = 0$.

Come trovo la base di soluzioni?

ES

$$u'' - 3u' + 2u = 0$$

$Du = u'$ $\quad D$ = operatore derivata.

$$D : C^\infty \to C^\infty$$

$$D^2 u - 3 D^1 u + 2 D^0 u = 0$$

$D^2 = D \circ D \qquad D^1 = D \qquad D^0 = Id.$

$$\left(D^2 - 3D^1 + 2D^0\right) u = 0$$

$$L = D^2 - 3D + 2$$
$$= P(D)$$

$$P(t) = \underbrace{t^2 - 3t + 2}$$
$$= (t-2)\cdot(t-1)$$

$$L = (D-2)\cdot(D-1)$$

$$(D-2)\underbrace{(D-1)\,u} = 0$$

$$(D-1)u = u' - u = 0$$

$$u(x) = \underbrace{c\cdot e^{x}}$$

$$(D-1)(D-2)u = 0$$

$$(D-2)u = 0$$

$$u' = 2u$$

$$u(x) = c\cdot e^{2x}$$

$$Du = \lambda u \Rightarrow \underline{\underline{u = e^{\lambda x}}}$$

Tutte le soluzioni sono:

$$u(x) = A \cdot e^{x} + B e^{2x}$$

$\square$

Se $L : V \to V$ <u>lineare</u>

$$Lu = 0$$

se $Lu_1 = 0$

e $Lu_2 = 0$

$$L(\lambda_1 u_1 + \lambda_2 u_2)$$

$$= \lambda_1 \underbrace{L u_1}_{= 0} + \lambda_2 \underbrace{L u_2}_{= 0} = 0$$

$u_1 = e^x \qquad u_2 = e^{2x}$

sono indipendenti?

$u_2 = \lambda \cdot u_1 \qquad e^{2x} = \lambda e^x$

$\lambda \in \mathbb{R}$

$$\Downarrow$$

$e^x = \lambda \in \mathbb{R}$

assurdo.

sì

$W = \text{span}\{u_1, u_2\} \qquad \dim W = 2$

$W \subseteq \ker L \qquad \dim \ker L = 2$

teorema astratto

$$\Downarrow$$

$W = \ker L$.

$(t-1)^2$

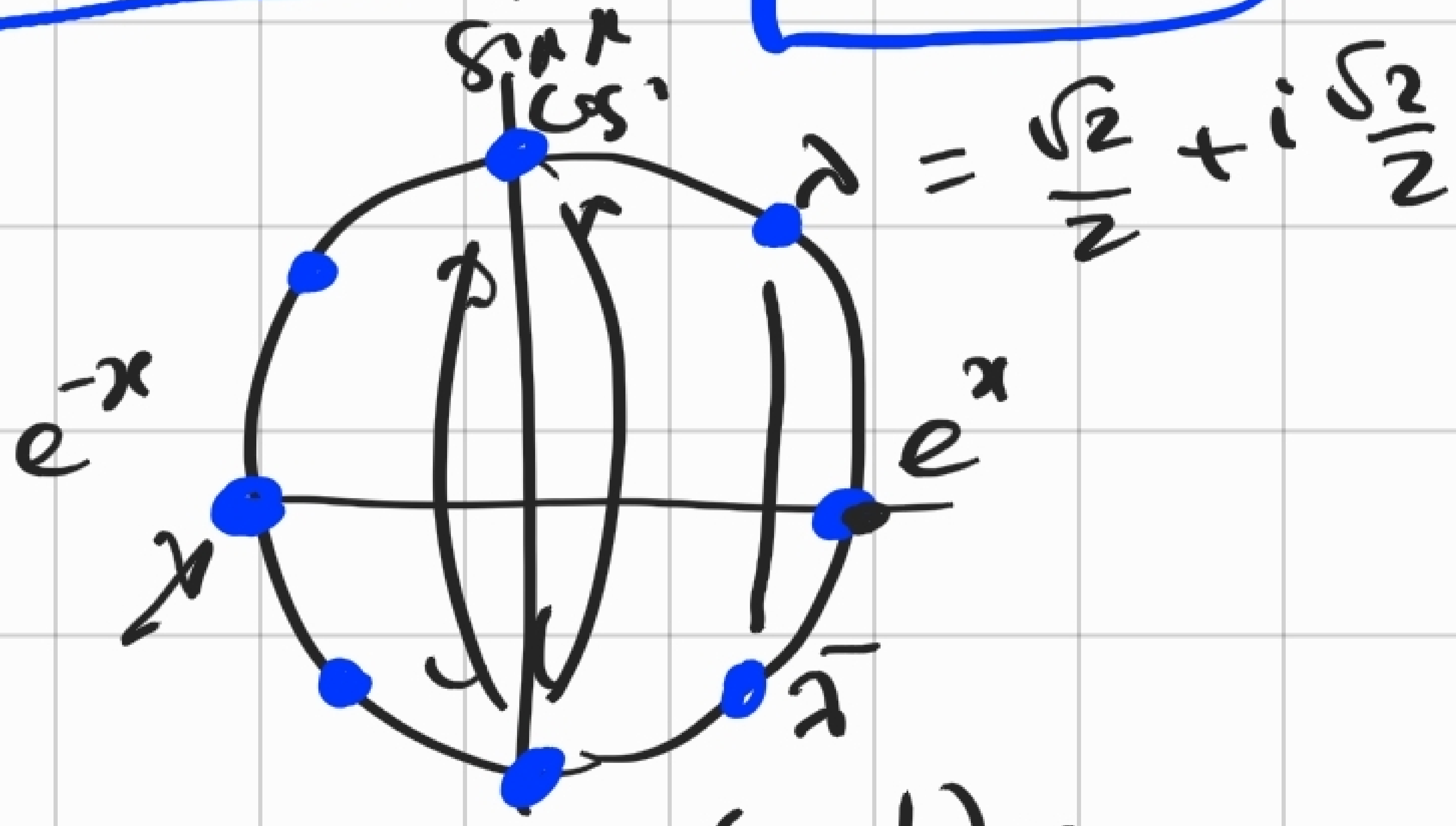
$u'' - 2u' + u = 0$

$u^{(8)} = u$

$\boxed{u''' = u}$

$z^8 - 1 = 0$

$\boxed{z^8 = 1}$



$\lambda = a \pm ib$

$e^{(a+ib)x} = e^{x} \cdot e^{ibx}$

$\cos bx + i \operatorname{sen} bx$

$e^{ax} \cos(bx)$

$e^{ax} \sin(bx)$

$\boxed{u = e^{\frac{\sqrt{2}}{2}x} \cdot \cos \frac{\sqrt{2}}{2} x}$