ANALISI MATEMATICA B
LEZIONE 52 - 12.2.2021

$$\lim_{x \to \infty} x - x^2\left(\ln\left(1+\sin\frac{1}{x}\right)\right)$$

$$y = \frac{1}{x} \qquad \begin{matrix} y \to 0 \\ \sin y = y - \frac{y^3}{6} + o(y^4) \end{matrix}$$

$$\ln(1+\sin y) = \ln\left(1 + y - \frac{y^3}{6} + o(y^4)\right)$$

$$= \underbrace{y - \frac{y^3}{6} + o(y^4)}_{t} + o(y) = y + o(y).$$

$$\ln(1+t) = t + o(t)$$

$$\boxed{\lim_{x \to 0} x^2 \cdot \ln\left(1+\sin\frac{1}{x}\right) \qquad \not{A}}$$

Formula di Taylor con il resto di Lagrange

$f \in C^{n+1}(I)$, I intervallo, $x_0 \in I$

P polinomio di Taylor d'ordine n per f centrato in x_0:

Allora $\forall x \in I$, $x \neq x_0$, $\exists\, y \in \begin{cases} (x_0, x) & \text{se } x > x_0 \\ (x, x_0) & \text{se } x < x_0 \end{cases}$ t.c.

$$f(x) = P(x) + \frac{f^{(n+1)}(y)}{(n+1)!}(x - x_0)^{n+1}$$

dim $\underline{x > x_0}$

$$\frac{f(x) - P(x)}{(x - x_0)^{n+1}} = \frac{[f(x) - P(x)] - \overbrace{[f(x_0) - P(x_0)]}^{=0}}{(x - x_0)^{n+1} - \underbrace{(x_0 - x_0)^{n+1}}_{=0}}$$

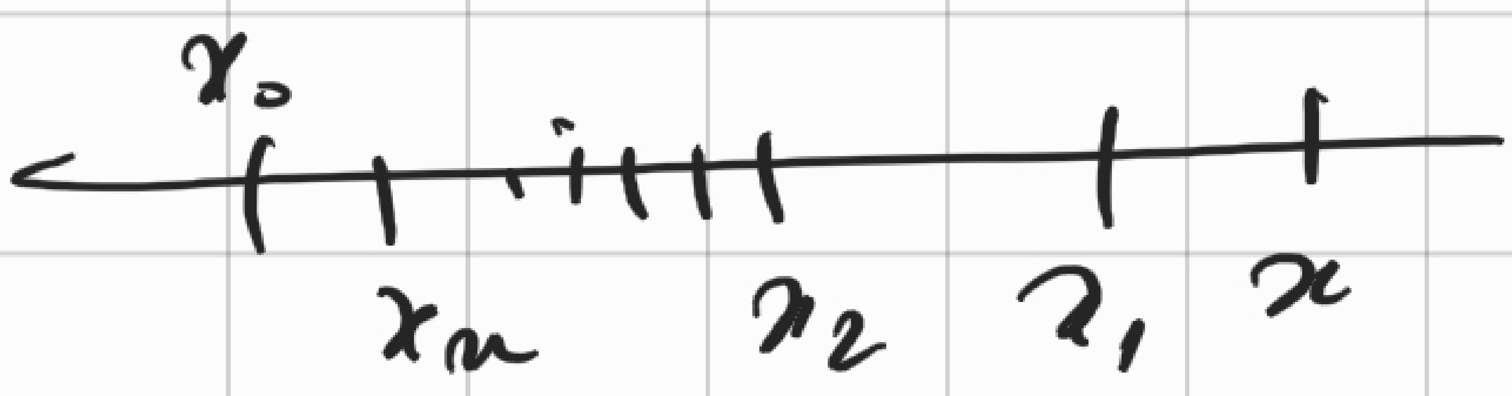
$\exists\, x_1 \in (x_0, x)$

$$\overset{\text{Cauchy}}{=} \frac{f'(x_1) - P'(x_1)}{(n+1)(x_1 - x_0)^n} = \frac{[f'(x_1) - P'(x_1)] - \overbrace{[f'(x_0) - P'(x_0)]}^{=0}}{(n+1)(x_1 - x_0)^n - \underbrace{(n+1)(x_0 - x_0)^n}_{=0}}$$

$\exists\, x_2 \in (x_0, x_1)$

$$\overset{\text{Cauchy}}{=} \frac{f''(x_2) - P''(x_2)}{(n+1) \cdot n\, (x_2 - x_0)^{n-1}} = \cdots$$

itero n volte



$$= \frac{f^{(n)}(x_n) - P^{(n)}(x_n)}{(n+1)\, n\, (n-1) \ldots 2 \cdot (x_n - x_0)}$$

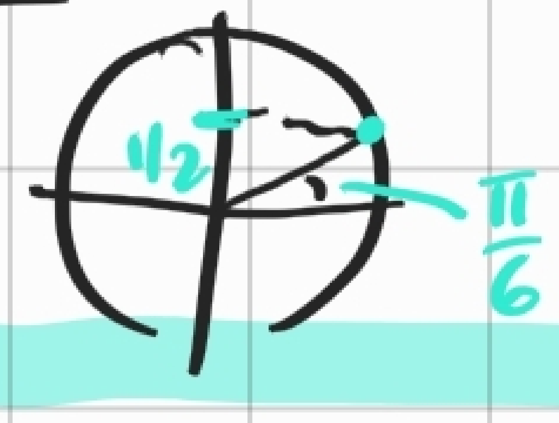
$$\overset{\text{Cauchy}}{=} \frac{f^{(n+1)}(x_{n+1}) - \overbrace{\cancel{P^{(n+1)}(x_{n+1})}}^{=0}}{(n+1)!} \qquad \square$$

$y = x_{n+1}$

$$\left[\quad P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \right]$$

$\lfloor$ Se $\deg P \le n \quad P^{(n+1)} = 0 \rfloor$

Esempio $\quad \arcsin x = \underbrace{\underbrace{x + \frac{1}{2} \cdot \frac{x^3}{3}} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5}} + o(x^5)$



$$\boxed{f(x) = \arcsin x = x + \frac{x^3}{6} + \frac{f^{IV}(y)}{4!} x^4}$$

$x_0 = 0 \qquad x > 0$

$x_0 = 0$

$y \in \underbrace{(0, x)}$

$$\pi = 6 \cdot \frac{\pi}{6} = 6 \arcsin \frac{1}{2} = 6 \cdot \left[\frac{1}{2} + \frac{1}{6 \cdot 2^3} + \frac{f^{IV}(y)}{24} \frac{1}{2^4} \right]$$

$$= 3 + \frac{1}{8} + \frac{f^{IV}(y)}{24 \cdot 16} = \underset{\uparrow}{3{,}125} + \frac{f^{IV}(y)}{24 \cdot 16}$$

VERSIONE CORRETTA

$$f(x) = \arcsin x$$

$$f'(x) = (1-x^2)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = (1-x^2)^{-3/2} \cdot x$$

$$f'''(x) = 3 \cdot (1-x^2)^{-5/2} x^2 + (1-x^2)^{-3/2}$$

$$f^{IV}(x) = -3 \cdot \frac{5}{2}(1-x^2)^{-7/2}(-2x) \cdot x^2 + 3(1-x^2)^{-\frac{5}{2}} 2x - \frac{3}{2}(1-x^2)^{-\frac{5}{2}}(-2x)$$

$$\nearrow \quad = 15(1-x^2)^{-7/2} \cdot x^3 + 9(1-x^2)^{-5/2} \cdot x$$

$$= (1-x^2)^{-7/2} x \left[15x^2 + 9(1-x^2)\right] = \frac{x \cdot (6x^2+9)}{(1-x^2)^{7/2}}$$

Se $x \in (0, \frac{1}{2}) \qquad |f^{IV}(x)| \le \qquad \nearrow \quad \sqrt{2} \in (1.41, 1.42)$

$x^2 \in (0, \frac{1}{4}) \qquad \nearrow$

$$1-x^2 \in \left(\frac{3}{4}, 1\right) \qquad \underbrace{(1-x^2)^{-\frac{7}{2}}} \in \left(1, \left(\frac{4}{3}\right)^{7/2}\right) \qquad 7/2 < 4$$

$$(1-x^2)^{-1} \in \left(1, \frac{4}{3}\right) \qquad \subseteq \left(1, \frac{256}{81}\right) \subseteq \underbrace{(1,4)}$$

$$6x^2 \in \left(0, \frac{3}{2}\right) \qquad 6x^2+9 \in (9, 11)$$

$$x \cdot (6x^2+9) \in \left(\frac{9}{2}, \frac{11}{2}\right) \subseteq (4, 6)$$

$$f^{IV}(x) \in (4, 24) \qquad \cancel{x}\ x \in \left(0, \frac{1}{2}\right)$$

$$\frac{f^{IV}(y)}{4!\,2^4} \in \left(\frac{4}{4!\,2^4}, \frac{\cancel{24}}{\cancel{4!}\cdot 2^4}\right)$$

$1:16 = 0{,}0625$
100
40
80

$$16\times$$
$$6$$
$$\overline{96}$$

$$= \left(\frac{1}{6\cdot 16}, \frac{1}{16}\right) \subseteq (0.01, 0.0625)$$

$$3{,}125$$
$$0{,}0625$$
$$\overline{3{,}1875}$$

$$\pi = 3.125 + \frac{f^{IV}(y)}{4!\,2^4} \in (3{,}135, 3.1875)$$

$$y = \sin t$$

Trucco:

$$f(\sin t) = t$$

$$f'(\sin t)\cos t = 1$$

$$f'(\sin t) = \frac{1}{\cos t}$$

$$f''(\sin t)\cos t = \frac{\sin t}{\cos^2 t}$$

$$f''(\sin t) = \sin t \cdot \cos^{-3} t$$

$$f'''(\sin t)\cos t = \cos^{-2} t + 3\sin^2 \cos^{-2}$$

$$\vdots$$

$\triangleright$

Serie di Taylor per f centrata in x_0:

$$g(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

ES se

$$f(x) = \frac{1}{1+x} \qquad f: \mathbb{R} \setminus \{-1\} \to \mathbb{R}$$

$$g(x) = \sum_{k=0}^{+\infty} (-1)^k x^k = \sum_{k=0}^{+\infty} (-x)^k$$

converge se $|x|<1$

$$= \frac{1}{1+x} = f(x)$$

Non sempre la serie di Taylor converge al valore della funzione:

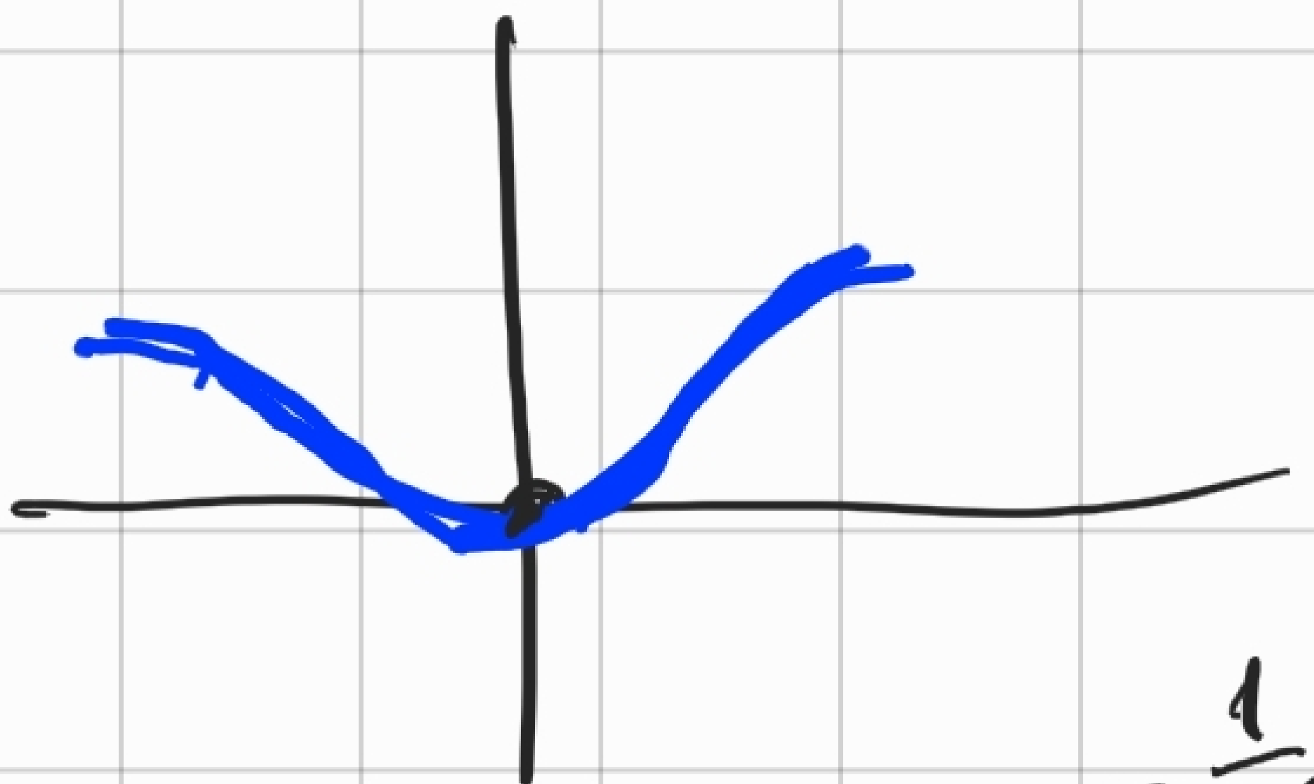
Es $f \in C^{\infty}(\mathbb{R})$

$$f^{(k)}(0) = 0 \quad \forall k \in \mathbb{N}$$

$$\sum \frac{f^{(k)}(0)}{k!} x^k = \sum 0 \cdot x^k = 0$$

Non è analitica.

Ma $f(0) = 0$ ma $f(x) > 0$ $\forall x \neq 0$.



$$\Downarrow$$

$$f(x) = o(x^n) \quad \forall n.$$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

per $x \to 0$ $\quad \frac{1}{x^2} \to +\infty$

$$e^{\frac{1}{x^2}} >> \left(\frac{1}{x^2}\right)^n \qquad \forall n$$

$$e^{-\frac{1}{x^2}} << x^{2n} \qquad \forall n$$

$$e^{-\frac{1}{x^2}} \in o\left(x^n\right) \quad \forall n$$

$$P_n = 0 \qquad \forall n. \quad \dots \dots$$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

se $x \neq 0$ $\quad \underbrace{f'(x) = 2x^{-3} \cdot e^{-\frac{1}{x^2}}}$ $\qquad \boxed{e^{-\frac{1}{x^2}} << x^3}$

$$\left.\begin{array}{l} \lim_{x \to 0} f'(x) = 0 \\ f \text{ continua} \end{array}\right] \Rightarrow f'(0) = 0$$

f derivabile in 0.

se $x \neq 0$

$$f^{(k)}(x) = P_k\left(\frac{1}{x}\right) \cdot e^{-\frac{1}{x^2}} \to 0 \text{ per } x \to 0$$

Def diciamo che f è analitica $f: A \to \mathbb{R}$ se f è C^∞ e se $\forall x_0 \in A$ tale che la serie di Taylor di f ha R di convergenza positivo e la sua somma coincide con f.

Teorema (criterio di analiticità)
se $f \in C^\infty$ e se $\forall x_0$ $\exists \rho > 0$ $\exists M, L > 0$ t.c. $\forall x \in \underbrace{(x_0 - \rho, x_0 + \rho)}$ si ha:

$$\boxed{|f^{(n)}(x)| \leq M \cdot L^n n!}$$

Allora f è analitica.

dim

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \underbrace{\frac{f^{(n+1)}(y)}{(n+1)!}(x - x_0)^{n+1}}$$

Se $|x - x_0| < \rho \Rightarrow |y - x_0| < \rho$

$$\left| \frac{f^{(n+1)}(y)}{(n+1)!}(x - x_0)^{n+1} \right| \leq \frac{M L^{n+1} \cancel{(n+1)!}}{\cancel{(n+1)!}} |x - x_0|^{n+1}$$

$$= M(Lp)^{n+1} \xrightarrow[n \to +\infty]{} 0$$

Se p è abbastanza piccolo $Lp < 1$

$$\sum_k a_k \qquad k \to +\infty$$

$$a_k = \ln \cos \frac{1}{k}$$

$$= f\left(\frac{1}{k}\right)$$

$x = \frac{1}{k}$ $\quad x \to 0$

$$f(x) = \ln \cos x$$

Es $\displaystyle\sum_{k=1}^{+\infty} \ln \cos \frac{1}{k}$

$$\boxed{\ln(1+y) = y + o(y)}$$

$$\ln \cos x = \ln\left(1 - \frac{x^2}{2} + o(x^2)\right)$$

$$= -\frac{x^2}{2} + o(x^2) + \cancel{o(x^2)}$$

per $x \to 0$

$x = \frac{1}{k}$

$k \to +\infty$

$$\ln \cos \frac{1}{k} = -\frac{1}{2k^2} + o\left(\frac{1}{k^2}\right) \text{ per } k \to +\infty$$

$$\boxed{\ln \cos \frac{1}{k} \sim -\frac{1}{2k^2}}$$

$$\frac{\ln \cos \frac{1}{k}}{-\frac{1}{2k^2}} = \frac{-\frac{1}{2k^2} + o\left(\frac{1}{k^2}\right)}{-\frac{1}{2k^2}}$$

$$= 1 + \cancel{\frac{o\left(\frac{1}{k^2}\right)}{-\frac{1}{2}\frac{1}{k^2}}}^{\,0} \to 1$$

$\sum \ln \cos \frac{1}{k}$ ha lo stesso carattere

di $-\frac{1}{2}\sum \frac{1}{k^2} < +\infty$.

$\cos x \sim 1$ per $x \to 0$

$\sin x \sim x$

$1 - \cos x \sim \frac{x^2}{2}$