ANALISI MATEMATICA B
LEZIONE 39 - 13. 1. 2021

Convergenza condizionata / incondizionata.

$$\sum_{k=0}^{+\infty} a_k \overset{?}{=} \sum_{j=0}^{+\infty} a_{\sigma(j)}$$

$k = \sigma(j)$ $\qquad \sigma : \mathbb{N} \to \mathbb{N}$ bigettiva

Teorema (convergenza condizionata) $(a_k \in \mathbb{R})$
Se $\sum a_k$ converge ma non assolutamente allora $\forall S \in [-\infty, +\infty]$ $\exists \sigma : \mathbb{N} \to \mathbb{N}$ bigettiva tale che

$$\sum_{j=0}^{+\infty} a_{\sigma(j)} = S.$$

ES $\left\{ 1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, \dots \right\}$

dim $\sum$ termini positivi $= +\infty$
$\sum$ termini negativi $= -\infty$

(se fossero entrambe finite la serie convergerebbe assolutamente)

(se una sola fosse infinita la serie non convergerebbe semplicemente).

$$\overrightarrow{\qquad\qquad\qquad\qquad} S$$

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \uparrow - \frac{1}{2} - \frac{1}{4} \dots$$

$< S$

$\square$

<u>Teorema</u> (convergenza incondizionata)

($a_k \in \mathbb{C}$)

Se $\sum a_k$ converge assolutamente ($\sum |a_k| < +\infty$) e $\sigma : \mathbb{N} \to \mathbb{N}$ è una biiezione allora

$$\sum_{j=0}^{+\infty} a_{\sigma(j)} = \sum_{k=0}^{+\infty} a_k$$

$(k = \sigma(j))$

(Potrei scrivere $\sum_{k \in \mathbb{N}} a_k$.)

dim

Devo dimostrare che $\lim_{N \to +\infty} \sum_{j=0}^{N} a_{\sigma(j)} = \sum_{k=0}^{+\infty} a_k$

ovvero $\forall \varepsilon > 0 \ \exists \bar{d} : N > \bar{d} : \left| \sum_{j=0}^{N} a_{\sigma(j)} - \sum_{k=0}^{+\infty} a_k \right| < \varepsilon$

Dato $\varepsilon > 0 \ \exists M : \left| \sum_{k=0}^{M} a_k - \sum_{k=0}^{+\infty} a_k \right| < \frac{\varepsilon}{2}$

Basta mostrare che esiste $\bar{d}$: se $N > \bar{d}$

$$(*) = \left| \sum_{j=0}^{N} a_{\sigma(j)} - \cancel{\sum_{k=0}^{M} a_k} \right| \overset{?}{<} \frac{\varepsilon}{2}$$

$$\cancel{a_0} \quad \cancel{a_1} \quad \cancel{a_2} \quad \cancel{\dots} \quad \cancel{a_M} \quad | \quad a_{\sigma(0)} \quad \dots$$

$$a_{\sigma(0)} \quad a_{\sigma(1)} \quad \dots \dots \quad \cancel{a_1} \quad \dots \quad \cancel{a_0} \quad \dots \quad \cancel{a_2} \quad a_{\sigma(N)}$$

$$\sigma(\sigma^{-1}(0))$$

Per ipotesi $\sum |a_k|$ è convergente.

Per il teorema della coda:

$$\lim_{n \to +\infty} \sum_{k=n}^{+\infty} |a_k| = 0$$

$\forall \varepsilon > 0 \quad \exists M$ tale che $\displaystyle\sum_{k=M+1}^{+\infty} |a_k| < \frac{\varepsilon}{2}$

$$\cancel{A} \leq \sum_{k=n}^{+\infty} |a_k| < \frac{\varepsilon}{2} \qquad \square$$

Ho scelto $N > \alpha$

$$\underline{\{\sigma(0), \sigma(1), \sigma(2), \dots, \sigma(\alpha)\}}$$

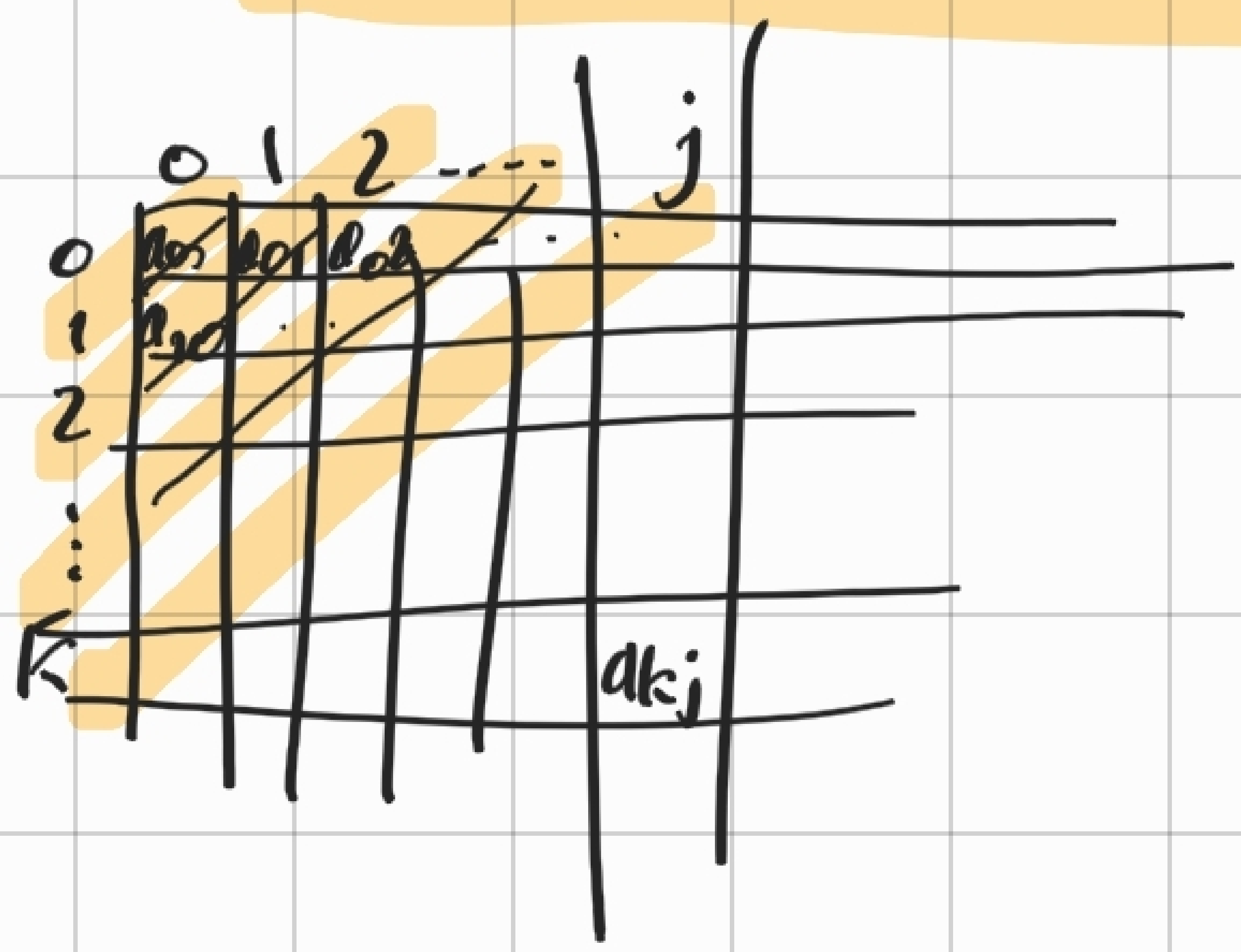
$$\supseteq \{0, 1, 2, \dots, N\}$$

$$\left(\alpha = \max \sigma^{-1}(\{0, 1, \dots, N\})\right)$$

Teorema (somme alla Cauchy) $a_{k,j} \in \mathbb{C}$

$k,j \in \mathbb{N}$ Se $\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}|a_{k,j}| < +\infty$

Allora: $$\boxed{\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} a_{k,j} = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_{k,n-k}}$$



$k+j=n$

$j=n-k$

[Idea: $\sum_{(k,j)\in\mathbb{N}\times\mathbb{N}} a_{k,j}$ è ben definita.]

dim Ipotesi $\sum_{k=0}^{+\infty}\underbrace{\sum_{j=0}^{+\infty}|a_{k,j}|} < +\infty$

In particolare $\forall k$ $\underbrace{\sum_{j=0}^{+\infty}|a_{k,j}|<+\infty}$

Notazione $b_k = \sum_{j=0}^{+\infty} a_{k,j}$

$$c_n = \sum_{k=0}^{n} a_{k,n-k}$$

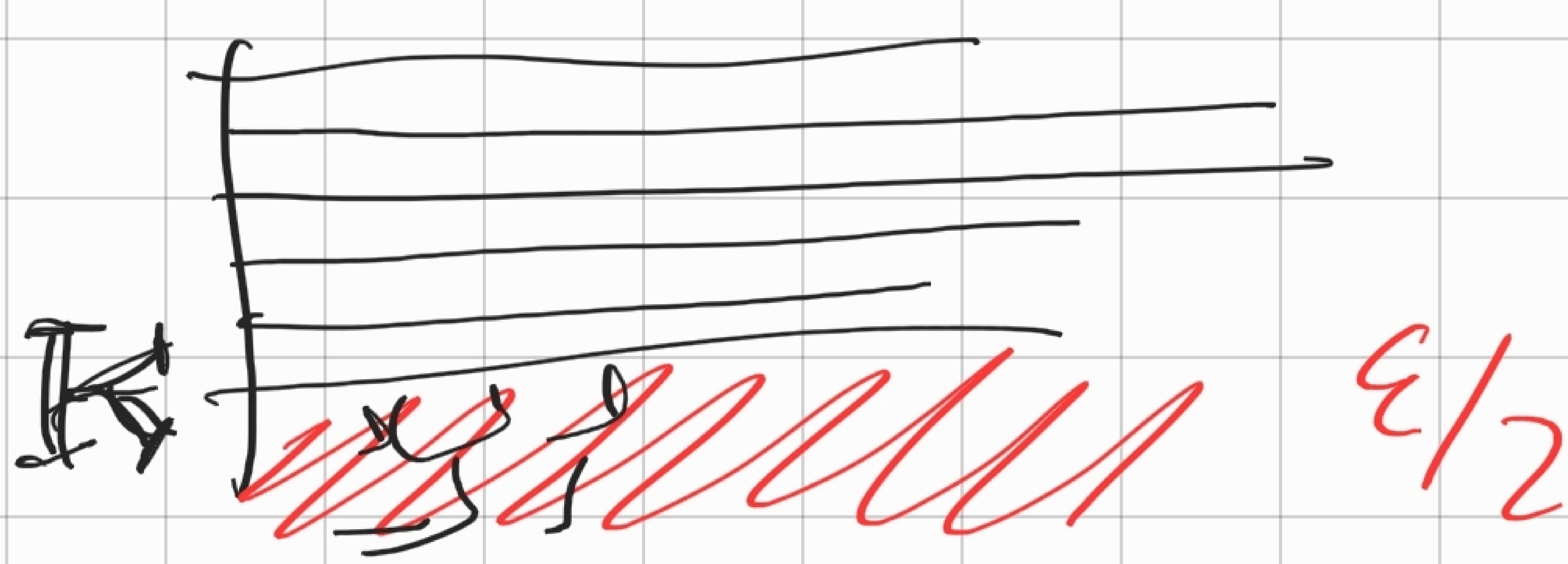
Tesi: $\displaystyle\sum_{k=0}^{+\infty} b_k = \sum_{n=0}^{+\infty} c_n$

cioè: $\forall \varepsilon > 0 \quad \exists \alpha : N > \alpha$

$$\left| \sum_{n=0}^{N} c_n - \sum_{k=0}^{+\infty} b_k \right| < \varepsilon$$

Per ipotesi $\displaystyle\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |a_{k,j}| < \varepsilon$

$$\exists K \quad \sum_{k=K+1}^{+\infty} \sum_{j=0}^{+\infty} |a_{k,j}| < \frac{\varepsilon}{2}$$



In particolare $\left|\sum_{k=K+1}^{+\infty} b_k\right|$

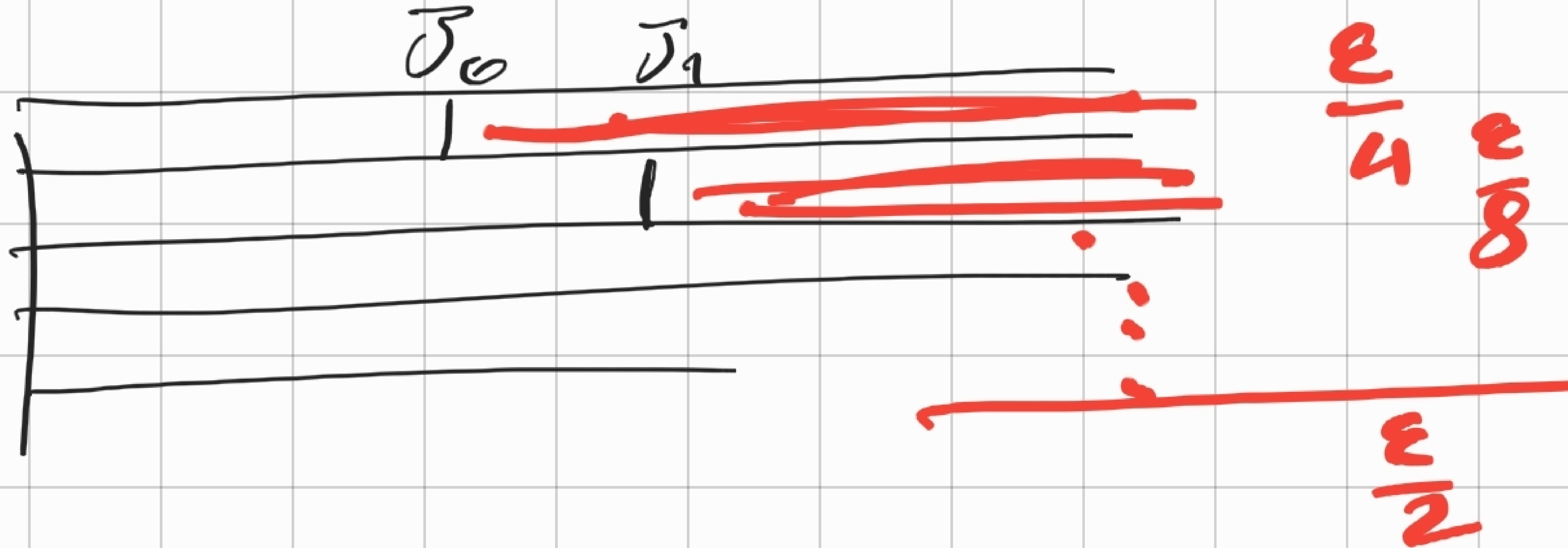
$$\le \sum_{k=K+1}^{+\infty} |b_k| = \sum_{k=K+1}^{+\infty} \left|\sum_{j=0}^{+\infty} a_{kj}\right|$$

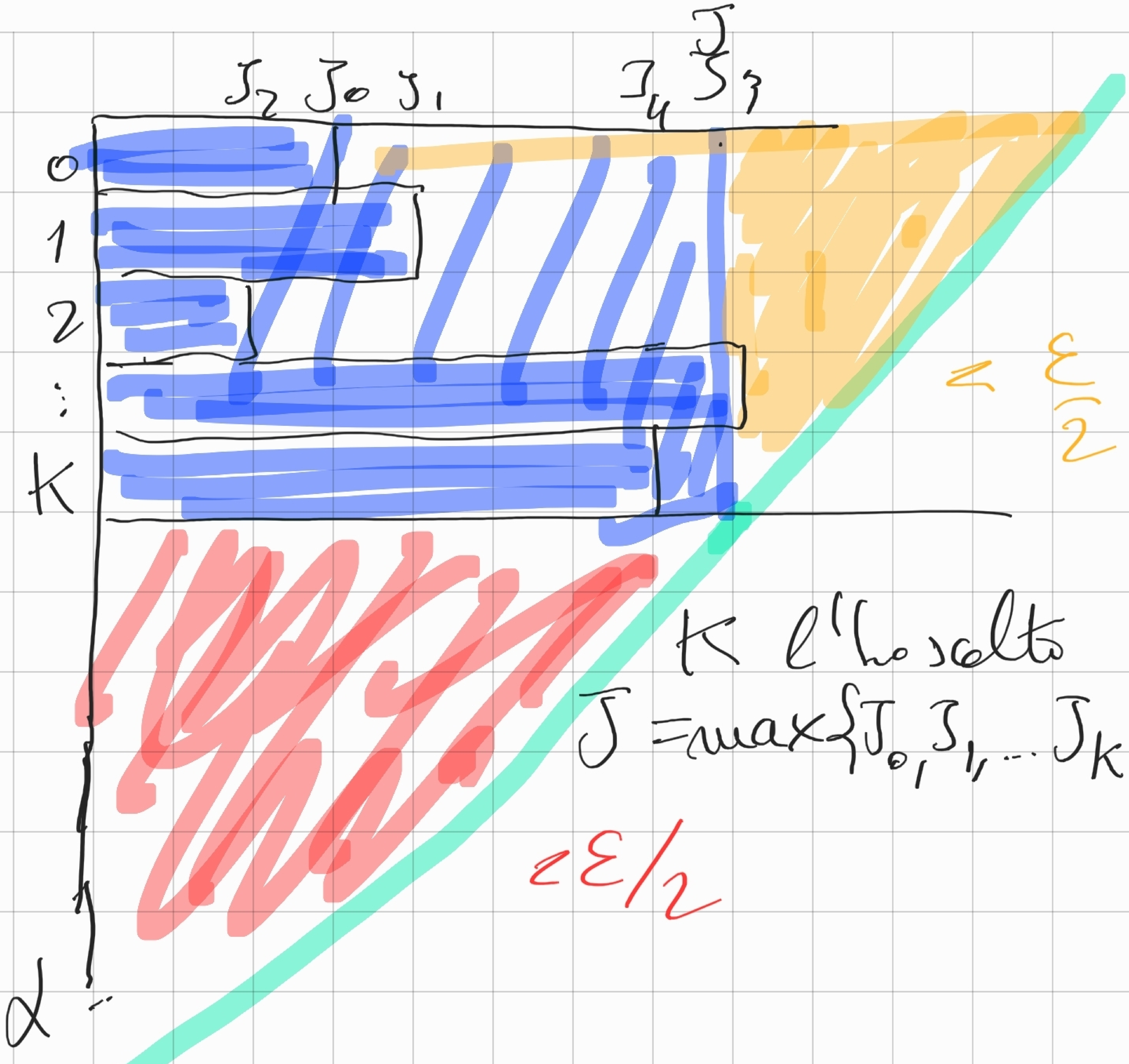
$$\le \sum_{k=K+1}^{+\infty} \sum_{j=0}^{+\infty} |a_{kj}| < \frac{\varepsilon}{2}$$

Per le righe: $\forall k$

$$\sum_{j=0}^{+\infty} |a_{kj}| < +\infty$$

$\forall k: \exists\ J_k : \quad \sum_{j=J_k+1}^{+\infty} |a_{kj}| < \frac{\varepsilon}{4\cdot 2^k}$





K l'ho scelto

$J = \max\{J_0, J_1, \dots J_K\}$

Prendo $N > \alpha = K + J$

$$\left| \sum_{n=0}^{N} c_n - \sum_{k=0}^{K} \sum_{j=0}^{J} a_{kj} \right| < \varepsilon$$

$$\left| \sum^{K} \sum^{J} a_{kj} - \sum b_k \right| < \varepsilon \quad \square$$

Esponenziale complesso

$$\exp(z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!}$$

$$\exp(z+w) \overset{?}{=} \exp(z) \cdot \exp(w)$$

$$\boxed{\exp(z+w)} = \sum_{n=0}^{+\infty} \frac{(z+w)^n}{n!} =$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \boxed{\frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}} =$$

$$\left(\frac{1}{n!} \binom{n}{k} = \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{k!(n-k)!} \right)$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} a_{k,n-k}$$

posto $a_{k,j} = \frac{z^k}{k!} \cdot \frac{w^j}{j!}$

$$\exp(z)\cdot\exp(w) = \left(\sum_{k=0}^{+\infty} \frac{z^k}{k!}\right)\cdot\left(\sum_{j=0}^{+\infty}\frac{w^j}{j!}\right)$$

$$= \sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty}\frac{w^j}{j!}\right)\frac{z^k}{k!}$$

Teorema sulle Cauchy

$$= \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\frac{z^k}{k!}\cdot\frac{w^j}{j!}$$

$$= \sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty} a_{k,j}.$$

$$\boxed{\exp(z+w) = \exp(z)\cdot\exp(w)}$$

Ne $$\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}|a_{kj}| < +\infty$$

$$\sum_{k=0}^{+\infty}\sum_{j=0}^{+\infty}\left|\frac{z^k}{k!}\right|\cdot\left|\frac{w^j}{j!}\right|$$

$$= \sum_{k=0}^{+\infty}\left[\frac{|z|^k}{k!}\underbrace{\sum_{j=0}^{+\infty}\frac{|w|^j}{j!}}_{\exp(|w|)}\right]$$

$$= \sum_{j=0}^{+\infty}\frac{|w|^j}{j!}\sum_{k=0}^{+\infty}\frac{|z|^k}{k!} < +\infty$$

$\square$

(0) $\exp : \mathbb{C} \to \mathbb{C}$ è continua

(1) $\exp(0) = 1$

(2) $\exp(\bar{z}) = \overline{\exp(z)}$

(3) $\exp(z+w) = \exp(z) \cdot \exp(w)$

(4) $\forall z \in \mathbb{C}$ $\exp(z) \neq 0$ e

$$\exp(-z) = \frac{1}{\exp(z)}$$

$$1 = \exp(0) = \exp(z-z) = \exp(z) \cdot \exp(-z)$$

(5) $\displaystyle\lim_{z\to 0} \frac{\exp(z) - \overbrace{1}^{= \exp(0)}}{z} = 1$

$$\frac{\exp(z)-1}{z} = \frac{\sum_{k=0}^{+\infty} \frac{z^k}{k!} - 1}{z} = \frac{\sum_{k=1}^{+\infty} \frac{z^k}{k!}}{z}$$

$$= \sum_{k=1}^{+\infty} \frac{z^{k-1}}{k!} = \underbrace{\sum_{k=0}^{+\infty} \frac{z^k}{(k+1)!}}_{g(z)} \xrightarrow{z \to 0} g(0) = \frac{1}{1!} = 1$$

serie di potenze $R = +\infty$

$\to$ è continua in z $\square$

$$\lim_{z \to 0} g(z) = g(0)$$

A questo $\exp(x) = e^x$ $\forall x \in \mathbb{R}$

$$\boxed{e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}}$$

$$\frac{a^x - 1}{x} \to 1$$

$\forall x \in \mathbb{R}$: $\exp(x) = \boxed{a^x}$

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f

$$\sum_A f = \lim_{B \to A} \sum_B f$$

f A B finite

biiettiva

$$\sigma : A \to A$$

$$\downarrow f \quad \downarrow f$$

$$\mathbb{C} \quad \mathbb{C}$$

$$\sum_A f \circ \sigma = \sum_A f$$

$f \circ \sigma$

$\sigma(B)$