ANALISI MATEMATICA B
LEZIONE 9 (bis) – 12.10.2020

$n \cdot x \quad n \in \mathbb{N}, \; x \in \mathbb{R}$

$$\begin{cases} 0 \cdot x = 0 \\ (n+1) \cdot x = n \cdot x + x \end{cases}$$

$0 \cdot x = 0$

$1 \cdot x = 0 + x$

$2 \cdot x = 0 + x + x$

$\vdots$

m^n

$$\begin{cases} m^0 = 1 \\ m^{n+1} = m \cdot m^n . \end{cases}$$

$m^0 = 1$

$m^1 = 1 \cdot m$

$m^2 = 1 \cdot m \cdot m$

$\vdots$

In generale data $g : A \to A$

$g^2(x) = g(g(x))$

$g^3(x) = g(g(g(x)))$

$\vdots$

$g^n(x) = \underbrace{g(\ldots(g(x))\ldots)}_{n \text{ volte}}$

$g^2 = g \circ g$

$g^3 = g \circ g \circ g$

$g^n = \underbrace{g \circ \ldots \circ g}_{n \text{ volte}}$

$$g^1(x) = g(x)\ ,\quad g^0(x) = x$$

Definiamo l'iterata n-esima di g come segue:

$$\begin{cases} g^0(x) = x \\ g^{n+1}(x) = g\left(g^n(x)\right) \end{cases}$$

Proprietà dell'iterata:

$$g^m \circ g^n = g^{m+n}$$

$$\underbrace{\underbrace{(g \circ \cdots \circ g)}_{m} \circ \underbrace{(g \circ \cdots \circ g)}_{n}}_{m+n}$$

$(f \circ g) \circ h = f \circ (g \circ h)$

$f(g(h(x)))$

Esercizio $\forall m,n \in \mathbb{N}: \boxed{g^n \circ g^m = g^{m+n}}$

$P(n)$: $\quad \forall m \in \mathbb{N} \quad g^n \circ g^m = g^{m+n}$

(i) $P(0)$: $\quad g^0 \circ g^m \overset{?}{=} g^{m+0}$

$$\underbrace{g^0 \circ g^m}_{= g^m} = g^{m+0} \qquad \underline{\underline{ok}}$$

(ii) $P(n) \Rightarrow P(n+1)$

$P(n+1)$: $\quad g^{n+1} \circ g^m \overset{?}{=} g^{m+n+1}$

$$g \circ g^n \circ g^m = g \circ g^{m+n}$$

$$\Uparrow$$

$P(n)$: $\quad g^n \circ g^m = g^{m+n}$

Similmente si dimostra:

$$\left(g^m\right)^n = g^{n \cdot m}.$$

$$\underbrace{\left(\overbrace{g \circ \dots \circ g}^{m}\right) \circ \left(\overbrace{g \circ \dots \circ g}^{m}\right) \dots\dots \circ \left(\overbrace{g \circ \dots \circ g}^{m}\right)}_{n}$$

Applicando queste proprietà alla moltiplicazione si ottiene:

$$m \cdot a + n \cdot a = (m+n) \cdot a \quad \text{(proprietà distributiva)}$$

$$m \cdot (n \cdot a) = (m \cdot n) \cdot a \quad \text{(proprietà associativa)}.$$

Vale anche $n(a+b) = n \cdot a + n \cdot b$
(per induzione)

<u>Per le potenze</u>:

$$m^n \cdot m^k = m^{n+k}$$

$$\left(m^n\right)^k = m^{kn}.$$

(Se $m, n \in \mathbb{N}$ $\quad m \cdot n = n \cdot m$ —)
$\quad m + n = n + m$

Se $x \geq 0$, $n \geq m$, $n, m \in \mathbb{N}$

$$n \cdot x \geq m \cdot x$$

Esercizi sul principio di induzione $P(n)$

Esercizio dimostrare che $\forall n \in \mathbb{N}: n^n \geq n!$

$$n! = \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}_{n \text{ fattori}}$$

$$\begin{cases} 0! = 1 \\ (n+1)! = (n+1) \cdot n! \end{cases}$$

DEFINIZIONE RICORSIVA

$$n^n = \overbrace{n \cdot n \cdot \ldots \cdot n}^{n}$$

$$n! = \underbrace{1 \cdot 2 \cdot \ldots \cdot n}_{n}$$

Per induzione: (i) $P(0)$: $\quad 0^0 \geq 0!$ ✓

$$0^0 = 1, \quad 0! = 1$$

(Se non vi convince:

$P(1)$: $\quad 1^1 \geq 1!$

$$1^1 = 1, \quad 1! = 1$$

(ii) $P(n) \Rightarrow P(n+1)$

$P(n)$ ↑ ipotesi induttiva

$P(n+1)$: $\quad (n+1)^{n+1} \overset{?}{\geq} (n+1)!$ ok

$$(n+1)^{n+1} = (n+1) \cdot (n+1)^n \geq (n+1)\, n^n \geq (n+1) \cdot n! = (n+1)!$$

$\Uparrow$

$P(n)$: $\quad n^n \geq n!$ $\square$

Esercizio per quali $n \in \mathbb{N}$: $2^n \geq n^2$?

esponenziale — potenza

n	2^n	n^2
0	1	0
1	2	1
2	4	4
~~3~~	~~8~~	~~9~~
4	16	16
5	32	25
6	64	36
⋮		

Proviamo a dimostrare per induzione che $\forall n \geq 4$: $2^n \geq n^2$.

$Q(n)$: $P(n+4)$

Basta dimostrare $\forall n$: $Q(n)$

(i) $P(4)$: $2^4 \overset{?}{\geq} 4^2$, $2^4 = 16$, $4^2 = 16$

(ii) $P(n) \overset{?}{\Rightarrow} P(n+1)$ se $n \geq 3$.

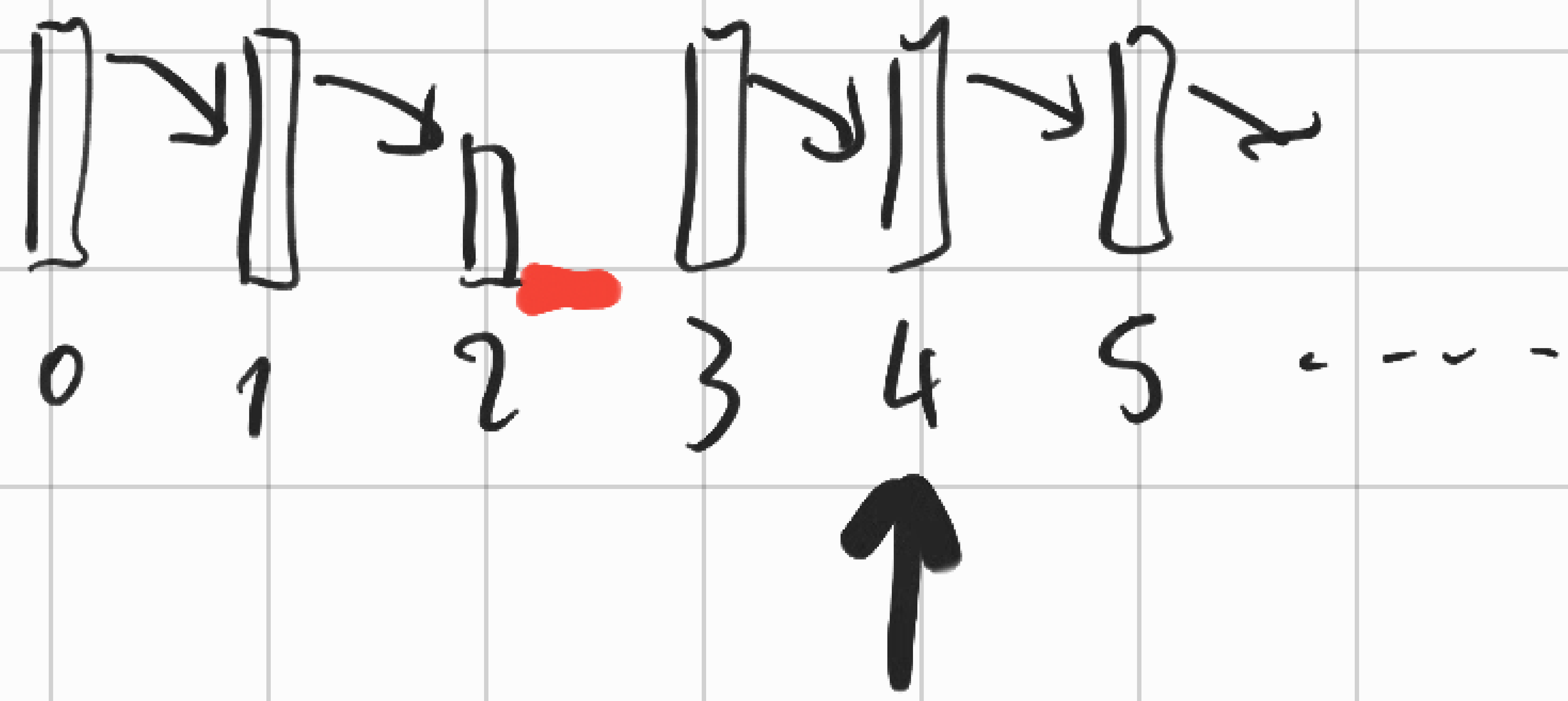
$P(n+1)$: $2^{n+1} \overset{?}{\geq} (n+1)^2 = n^2 + 2n + 1$

$2^n \geq n^2$

$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot n^2 = n^2 + n^2$

$n^2 = n \cdot n \geq 3 \cdot n \geq 2n + n \geq 2n + 3 \geq 2n+1$

$\square$



Sommatoria (e produttorie)

Sia $f : \mathbb{N} \to \mathbb{R}$ $\quad (\text{o } f : \mathbb{N} \to \mathbb{N})$

$$\sum_{k=1}^{n} f(k) = \underbrace{f(1) + f(2) + f(3) + \dots + f(n)}_{n \text{ addendi}}$$

$$\prod_{k=1}^{n} f(k) = \underbrace{f(1) \cdot f(2) \cdot f(3) \cdot \dots \cdot f(n)}_{n \text{ fattori}}$$

Formalmente $\quad \sum_{k=1}^{n} f(k) \leftarrow$ k è una variabile muta.

$$\begin{cases} \sum_{k=1}^{0} f(k) = 0 \\ \sum_{k=1}^{n+1} f(k) = \sum_{k=1}^{n} f(k) + f(n+1) \end{cases}$$

$$\begin{cases} \prod_{k=1}^{0} f(k) = 1 \\ \prod_{k=1}^{n+1} f(k) = \prod_{k=1}^{n} f(k) \cdot f(n+1) \end{cases}$$

Esempio $\sum_{k=1}^{4} (2k-1)$ $\qquad (f(k) = 2k-1)$

$$\sum_{k=1}^{4} (2k-1) = \sum_{k=1}^{3} (2k-1) + 7$$

$$= \sum_{k=1}^{2} (2k-1) + 5 + 7$$

$$= \sum_{k=1}^{1} (2k-1) + 3 + 5 + 7$$

$$= 1 + 3 + 5 + 7 = 16$$

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + 5 + 7 + \ldots + (2n-1)$$

n	$2n-1$	$\sum_{k=1}^{n}(2k-1)$
0		0
1	1	1
2	3	4
3	5	9
4	7	16
$\vdots$	$\vdots$	$\vdots$
n	$2n-1$	n^2 ?

← si potrebbe mostrare per induzione

<u>Es</u>: $n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n.$

Proprietà della sommatoria (linearità)

$$\rightarrow \sum_{k=1}^{n} \left(f(k)+g(k)\right) = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k).$$

$$\left(f(1)+g(1)\right)+\left(f(2)+g(2)\right)+\ldots+\left(f(n)+g(n)\right)$$

$$= \left(f(1)+\ldots+f(n)\right)+\left(g(1)+\ldots+g(n)\right)$$

$$\sum_{k=1}^{n} c \cdot f(k) = c \cdot \sum_{k=1}^{n} f(k)$$

$$c\cdot f(1) + c\cdot f(2) + \dots + c\cdot f(n) = c\cdot(f(1) + \dots + f(n))$$

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n}(-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1.$$

$$(a-b = a+(-b))$$

$$\left[-b = (-1)\cdot b \qquad (-n)\cdot x = -(n\cdot x)\right]$$

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + \dots + 1}_{n} = n\cdot 1 = n.$$

$$n\cdot x = \sum_{k=1}^{n} x$$

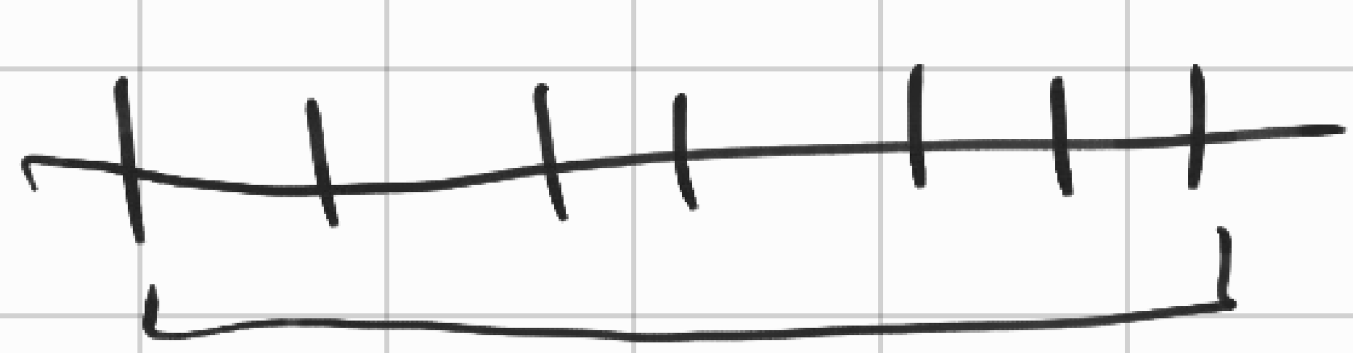
Si dimostrano per induzione

$$\sum_{k=1}^{n} k = \frac{n\cdot(n+1)}{2} \quad (??)$$

$a, b \in \mathbb{R}$

In generale:

$$\left[\sum_{k=1}^{n}(k\cdot a+b) = n\cdot\frac{(na+b+a+b)}{2}\right]$$



Nota: divisibile per 2??

Se $n, k \in \mathbb{Z}$ diremo che k divide n se $\exists\, m \in \mathbb{Z}$ t.c. $k \cdot m = n$.

In tal caso scriveremo $m = \frac{n}{k}$.

$$2 \cdot \sum_{k=1}^{n} k = n(n+1)$$

$$2 \cdot (1 + 2 + 3 + \dots + n)$$

$$\begin{aligned} = \ & 1 + 2 + 3 + \dots + (n-1) + n + \\ + \ & n + (n-1) + (n-2) + \dots + 2 + 1 \end{aligned}$$

$$\underbrace{(n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1)}_{n}$$

$$= n \cdot (n+1)$$