1. $f(x) = \begin{cases} \dfrac{e^{-\frac{1}{|x|}}}{|x|} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

La funzione è definita $\forall x \in \mathbb{R}$

$f(-x) = f(x) \Rightarrow f$ è pari

$\lim_{x \to \pm\infty} f(x) = 0 \Rightarrow y = 0$ è un asintoto orizzontale per $x \to +\infty$ e $x \to -\infty$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{x} = 0$$

$\Rightarrow f$ è continua su tutto $\mathbb{R}$.

Per $x > 0$

$$f'(x) = D\, \frac{e^{-\frac{1}{x}}}{x} = \frac{e^{-\frac{1}{x}} \cdot \frac{1}{x^2} \cdot x - e^{-\frac{1}{x}}}{x^2}$$

$$= e^{-\frac{1}{x}} \, \frac{\frac{1}{x} - 1}{x^2} = e^{-\frac{1}{x}} \, \frac{1 - x}{x^3}$$

$\lim_{x \to 0^+} f'(x) = 0 \Rightarrow \lim_{x \to 0^-} f'(x) = 0$ per simmetria $f'(-x) = -f'(x)$

$\Rightarrow f$ è derivabile in $x = 0$ e $f'(0) = 0$.

$\Rightarrow f$ è derivabile su tutto $\mathbb{R}$,

		-1		0		1	
f'	+	0	−	0	+	0	−
f	↗	max	↘	min	↗	max	↘

$f(0)=0$

$$f(\pm 1) = \frac{e^{-1}}{1} = \frac{1}{e}$$

$(0,0)$ minimo (assoluto) di f.

$\left(\pm 1, \frac{1}{e}\right)$ massimi (assoluti) di f.

Per $x>0$

$$f''(x) = D\, e^{-\frac{1}{x}}\frac{1-x}{x^3} = \frac{\left[e^{-\frac{1}{x}}\cdot\frac{1}{x^2}(1-x) - e^{-\frac{1}{x}}\right]x^3 - e^{-\frac{1}{x}}(1-x)3x^2}{x^6}$$

$$= \frac{e^{-\frac{1}{x}}\left[\left(\frac{1}{x^2}-\frac{1}{x}-1\right)x^3 - (1-x)3x^2\right]}{x^6}$$

$$= \frac{e^{-\frac{1}{x}}\left[x - x^2 - x^3 - 3x^2 + 3x^3\right]}{x^6}$$

$$= \frac{e^{-\frac{1}{x}}\left[2x^2-4x+1\right]}{x^5}$$

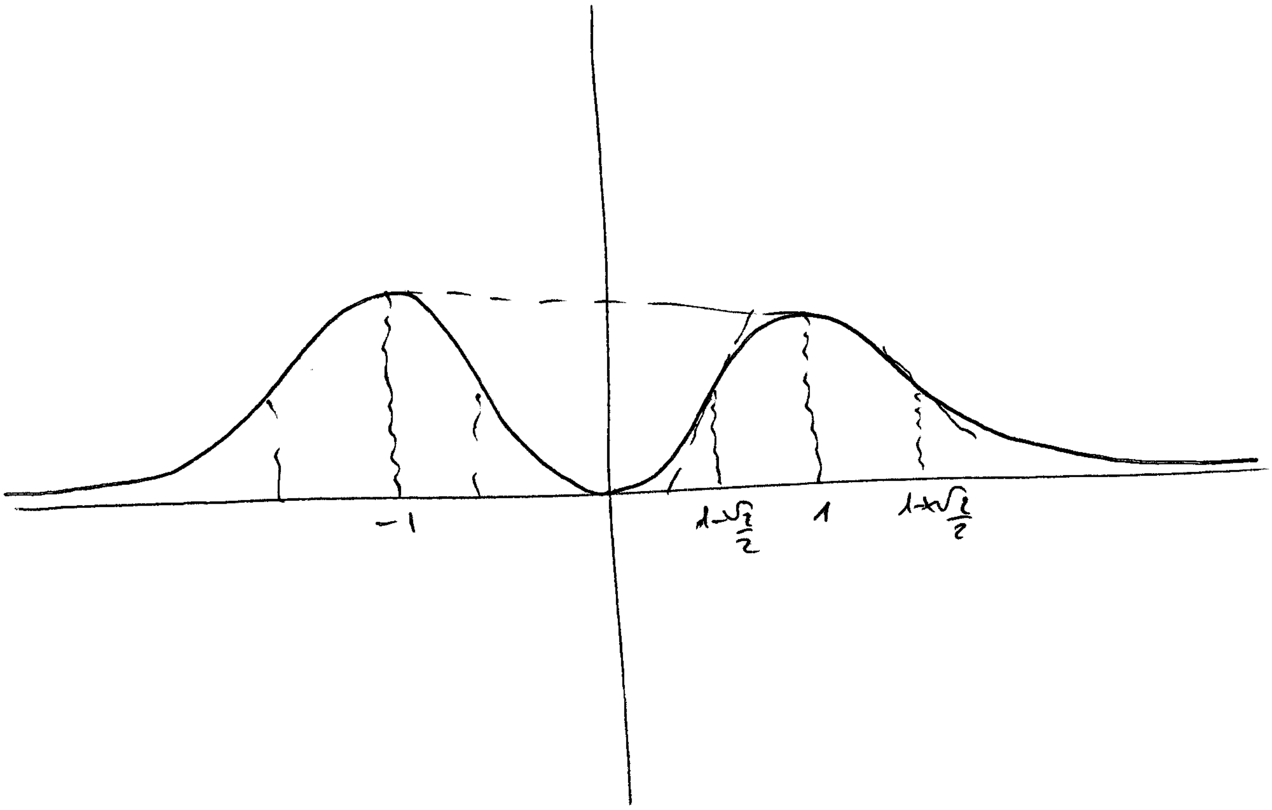
per simmetria: $f''(-x) = f''(x)$.

$2x^2-4x+1=0 \qquad x_{12} = \frac{2\pm\sqrt{4-2}}{2} = 1 \pm \frac{\sqrt{2}}{2}$

		$-1-\frac{\sqrt2}{2}$		$-1+\frac{\sqrt2}{2}$		0		$1-\frac{\sqrt2}{2}$		1		$1+\frac{\sqrt2}{2}$	
f''	$+$	0	$-$	0	$+$			0	$-$			0	$+$
f	$\smile$		$\frown$		$\smile$				$\frown$				$\smile$

$$f\left(\pm\left(1-\frac{\sqrt2}{2}\right)\right) = f\left(1-\frac{\sqrt2}{2}\right) = \frac{e^{-\frac{1}{1-\frac{\sqrt2}{2}}}}{1-\frac{\sqrt2}{2}} = \frac{e^{\frac{1}{\frac{\sqrt2}{2}-1}}}{1-\frac{\sqrt2}{2}} = \frac{e^{\frac{\frac{\sqrt2}{2}+1}{-1/2}}}{1-\frac{\sqrt2}{2}} = \frac{2e^{-\sqrt2-2}}{2-\sqrt2}$$

$$f\left(\pm\left(1+\frac{\sqrt{2}}{2}\right)\right) = f\left(1+\frac{\sqrt{2}}{2}\right) = \frac{e^{-\frac{1}{1+\frac{\sqrt{2}}{2}}}}{1+\frac{\sqrt{2}}{2}} = \frac{e^{-\frac{1-\frac{\sqrt{2}}{2}}{1/2}}}{1+\frac{\sqrt{2}}{2}} = \frac{2e^{\sqrt{2}-2}}{2+\sqrt{2}}$$



2)

$$\cos x = 1-\frac{x^2}{2}+o(x^2) \qquad e^x = 1+x+o(x)$$

$$e^{x^2} = 1+x^2+o(x^2) \qquad \log(1+x) = x+o(x)$$

$$\lim_{x\to 0}\left(\cos x\right)^{\frac{1}{\cos x - e^{(x^2)}}} = \lim_{x\to 0} e^{\frac{\log \cos x}{\cos x - e^{(x^2)}}}$$

$$= \lim_{x\to 0} e^{\frac{-\frac{x^2}{2}+o(x^2)}{\cancel{1}-\frac{x^2}{2}+o(x^2)-\cancel{1}-x^2+o(x^2)}}$$

$$= \lim_{x\to 0} e^{\frac{-\frac{x^2}{2}+o(x^2)}{-\frac{3}{2}x^2+o(x^2)}} = \lim_{x\to 0} e^{\frac{-\frac{1}{2}+o(1)}{-\frac{3}{2}+o(1)}}$$

$$= e^{\frac{-\frac{1}{2}}{-\frac{3}{2}}} = e^{\frac{1}{3}} = \sqrt[3]{e}.$$

3

$$\operatorname{tg} x = t$$
$$x = \operatorname{arctg} t$$
$$dx = \frac{1}{1+t^2}\,dt$$

$$\int \frac{1-\operatorname{tg}^4 x}{1+\operatorname{tg} x}\,dx$$

$$= \int \frac{1-t^4}{1+t}\,\frac{1}{1+t^2}\,dt$$

$$\left(\frac{1-t^4}{(1+t)(1+t^2)} = \frac{(1-t^2)\cancel{(1+t^2)}}{(1+t)\cancel{(1+t^2)}} = \frac{(1+t)(1-t)}{(1+t)} = 1-t \right)$$

$$= \int 1-t\,dt = t - \frac{t^2}{2} + c \,\Big|_{t=\operatorname{tg} x} = \operatorname{tg} x - \frac{\operatorname{tg}^2 x}{2} + c$$

4| Osserviamo che

$$|\sin \omega k + \cos \omega k| \le |\sin \omega k| + |\cos \omega k| \le 2$$

Dunque posto

$$a_k = (\sin \omega k + \cos \omega k)\left(\log(k+1) - \log k\right)^{1+\alpha}$$

si ha

$$|a_k| \le 2\left|\left(\log(k+1) - \log k\right)^{1+\alpha}\right|$$

$$\log(k+1) - \log k = \log\frac{k+1}{k} = \log\left(1+\frac{1}{k}\right) = \frac{1}{k} + o\left(\frac{1}{k}\right)$$

per $k \to +\infty$

$$|a_k| \le \left(\frac{1}{k} + o\left(\frac{1}{k}\right)\right)^{1+\alpha} = \left(\frac{1}{k}\right)^{1+\alpha} + o\left(\left(\frac{1}{k}\right)^{1+\alpha}\right)$$

ma $\sum \left(\frac{1}{k}\right)^{1+\alpha}$ converge se $\underline{\alpha > 0}$.

$\Rightarrow$ la serie data $\sum a_k$ converge assolutamente

$\Rightarrow$ converge.

$\omega = \pi$, $\alpha = 0$

$$\sin \omega k + \cos \omega k = \sin k\pi + \cos k\pi = 0 + (-1)^k$$

$$\sum a_k = \sum (-1)^k \log\left(1+\frac{1}{k}\right)$$

$\log(1+x)$ è crescente $\Rightarrow$ $\log\left(1+\frac{1}{k}\right)$ è decrescente

$$\lim_{k \to +\infty} \log\left(1+\frac{1}{k}\right) = 0$$

Per il criterio di Leibniz sulle serie a segni alterni la serie converge

<u>NB</u> ... non converge assolutamente in quanto

$$\sum \log\left(1+\frac{1}{k}\right) \sim \sum \frac{1}{k} = +\infty .$$