On Nonstandard Product Measure Spaces and Duality for Martingale Property

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Talk based on

1. J. Berger, H. Osswald, Y. Sun and J.-L. Wu: On nonstandard product measure spaces, *Illinois J. Math.*, **46** (2002), 319–330.

2. S. Albeverio, Y. Sun and J.-L. Wu: Martingale property of empirical processes, *Trans. Amer. Math. Soc.*, in press.

Outline

1. Doob's measurability problem

2. Loeb measure and rich measure spaces

3. Duality for martingale property

1 Doob's measurability problem

Start with the classical question in probability theory:

Can we speak of an uncountable number of equally weighted, independent random variables? (i.e., the index set must be a continuum set, e.g. $[0, 1]$ *)*

Since there is no uniform probability distribution on an infinitely countable set! Thus, the question above can be converted to

Is it possible to consider the concept of independence in the setting of a continuum of independent random variables?

Let $\Omega = \mathbb{R}^{[0,1]}$, the celebrated Kolmogorov extension theorem ensures that there exists a probability measure P on Ω constructed from probability distributions on $\mathbb{R}$ via project limit procedure.

Doob's observation (*Trans. Amer. Math. Soc.*, 1937):
For any $h \in \Omega$, the set

$$M_h := \{\omega \in \Omega : \omega(t) = h(t) \text{ except for countably many } t \in [0,1]\}$$

has P-outer measure 1. Now if h is non-Lebesgue measurable, then so is any $g \in M_h$, thus the set of non-Lebesgue measurable samples has P-outer measure 1! Doob then concludes

> "Processes with mutually independent random variables are only useful in the discrete parameter case"
>
> – J.L. Doob: *Stochastic Processes*, 1953.

Recent research of Yeneng Sun shows: No matter what kind of measure spaces are taken as the parameter and sample spaces of a stochastic process, independence and joint measurability with respect to the usual measure-theoretic product are never compatible with each other except for the trivial case where the random variables are essentially constant. Therefore, in order to study independence in the continuum setting, one has to go beyond the usual measure-theoretical framework!

So we come to the world of nonstandard analysis and let us recall briefly the structure of Loeb measure spaces.

2 Loeb measure and rich measure spaces

Loeb measure space

Let $(X, \mathcal{A}, \nu)$ be an internal measure space, that is,

- X is an internal set in the superstructure $V({}^*\mathbb{R})$
- $\mathcal{A} \subset \mathcal{P}(X)$ is an internal algebra
- $\nu : \mathcal{A} \rightarrow {}^*\mathbb{R}_+$ is a finitely additive internal measure.

Then the standard part ${}^\circ\nu : \mathcal{A} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ is finitely additive. Using Carathéodory extension principle, P. Loeb (*Trans. Amer. Math. Soc., 1975*) derived a standard measure space $(X, \mathcal{A}_L, \nu_L)$, the well-known Loeb measure space.

Products of measure spaces

Fix internal measure spaces $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$.

- $(T \times \Omega, \mathcal{T}_L \otimes \mathcal{A}_L, \lambda_L \otimes P_L)$ the usual product space;
- $(T \times \Omega, \mathcal{T} \boxtimes \mathcal{A}, \lambda \boxtimes P)$ the Loeb product space, which is obtained by taking Loeb measure space over the internal product $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \otimes P)$.

Some facts about these two products:

♢ R. Anderson, *Israel J. Math.* **25** (1976).

$$\mathcal{T}_L \otimes \mathcal{A}_L \subset \mathcal{T} \boxtimes \mathcal{A}, \quad \lambda \boxtimes P|_{\mathcal{T}_L \otimes \mathcal{A}_L} = \lambda_L \otimes P_L$$

♢ H.J. Keisler, *AMS Logic Colloquium* (1977). The Fubini property holds for Loeb product space.

♢ D. Hoover and D. Norman provided a specific example showing the inclusion can be proper.

♢ Y. Sun, *J. Math. Econ.* **29** (1998). The inclusion is strict iff both λ_L and P_L have non-atomic parts.

♢ H.J. Keisler and Y. Sun, *J. London Math. Soc.* (2004). $\lambda \boxtimes P$ is uniquely determined by λ_L and P_L.

$\diamondsuit$ Y. Sun, *Probab. Theory and Related Fields* **112** (1998). Pairwise independence and mutually independence are essentially equivalent.

$\diamondsuit$ Berger-Osswald-Sun-Wu, *Illinois J. Math.*, **46** (2002) The Loeb product $\mathcal{T} \boxtimes \mathcal{A}$ is very rich in the sense that there is a continuum of increasing Loeb product null sets with large gaps. Namely, If both λ_L and P_L are atomless, a class of sets $\{R_s \in \mathcal{T} \boxtimes \mathcal{A} : s \in [0,1]\}$ cab be constructed such that $\forall s \in [0,1]$,

- $\lambda \boxtimes P(R_s) = 0$;
- the outer measure $(\lambda_L \otimes P_L)^*(R_s) = 1$;
- $\forall s_1 < s_2, R_{s_1} \subset R_{s_2}$ and $(\lambda_L \otimes P_L)^*(R_{s_2} \setminus R_{s_1}) = 1$.

3 Duality for martingale property

As noted, the Loeb product probability spaces provide a suitable framework for the study of stochastic processes with independent random variables. We shall use this framework to consider a large collection of stochastic processes.

Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be two atomless Loeb probability spaces. Their usual measure-theoretic product is denoted by $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. (The completion of this product is denoted by the same notation.)

The Loeb product is denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Since $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ are assumed to be atomless, the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is very rich in the sense that it can be endowed with independent processes that are not measurable with respect to the usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$ but have essentially independent random variables with any variety of distributions. Thus, $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is always a proper extension of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ as shown above that there are many examples of Loeb product measurable sets that are not measurable in $\mathcal{I} \otimes \mathcal{F}$.

Keisler's Fubini theorem Let f be a real-valued integrable function on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Then

(i) for λ-almost all $i \in I$, $f(i, \cdot)$ is an integrable function on $(\Omega, \mathcal{F}, P)$;

(ii) the function $\int_\Omega f(i,\omega) dP(\omega)$ on I is integrable on $(I, \mathcal{I}, \lambda)$;

(iii)

$$\int_I \int_\Omega f(i,\omega) dP(\omega) d\lambda(i) = \int_{I\times\Omega} f(i,\omega) d\lambda \boxtimes P(i,\omega).$$

Similar properties hold for the functions $f(\cdot, \omega)$ on I and the function $\int_I f(i,\omega) d\lambda(i)$ on Ω.

Let T be a set of time parameters, which is assumed to be $\mathbb{N}$ or an interval (starting from 0) in the set $\mathbb{R}_+$ of non-negative real numbers. Let $\mathcal{B}(T)$ be the power set of T when T is $\mathbb{N}$, and the Borel σ-algebra on T when T is an interval. Let X be a real-valued measurable function on the mixed product measurable space $((I \times \Omega) \times T, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(T))$. We assume that for each $t \in T$, $X(\cdot, \cdot, t)$ is integrable on the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, i.e.,

$$\int_{I \times \Omega} |X(i, \omega, t)| d\lambda \boxtimes P(i, \omega) < \infty.$$

For any $i \in I$, let $X^i(\cdot,\cdot) := X(i,\cdot,\cdot)$ be the corresponding function on $\Omega \times T$; and for any $\omega \in \Omega$, let $X^\omega(\cdot,\cdot) := X(\cdot,\omega,\cdot)$ be the corresponding function on $I \times T$. Keisler's Fubini theorem implies that X^i is a measurable process on $(\Omega \times T, \mathcal{F} \otimes \mathcal{B}(T))$ for λ-almost all $i \in I$, and X^ω is a measurable process on $(I \times T, \mathcal{I} \otimes \mathcal{B}(T))$ for P-almost all $\omega \in \Omega$. Thus, X can be viewed as a family of stochastic processes, $X^i, i \in I$, with sample space $(\Omega, \mathcal{F}, P)$ and time parameter space T. For $\omega \in \Omega$, X^ω is called an empirical process with the index space $(I, \mathcal{I}, \lambda)$ as the sample space.

Note that we can take I to be a hyperfinite set in an ultrapower construction on the set of natural numbers, where I is simply an equivalence class of a sequence of finite sets. The cardinality of the set I in the usual sense is indeed the cardinality of the continuum. This means that $X^i, i \in I$ is indeed a continuum collection of stochastic processes.

For $i \in I$, let $\{\mathcal{F}_t^i\}_{t\in T}$ be a filtration on $(\Omega, \mathcal{F}, P)$. That is, it is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$ and each of them contains all the P-null sets in $\mathcal{F}$. The stochastic process X^i is said to be $\{\mathcal{F}_t^i\}_{t\in T}$-adapted if the random variable $X_t^i := X(i, \cdot, t)$ is $\mathcal{F}_t^i$-measurable for all $t \in T$. The X^i is said to be an $\{\mathcal{F}_t^i\}_{t\in T}$-martingale if it is $\{\mathcal{F}_t^i\}_{t\in T}$-adapted and

$$\mathbf{E}\left(X_t^i | \mathcal{F}_s^i\right) = X_s^i, \quad s, t \in T, s \leq t.$$

Let $\{\tilde{\mathcal{F}}_t^i\}_{t\in T}$ be the natural filtration generated by the stochastic process X^i as follows

$$\tilde{\mathcal{F}}_t^i := \sigma(\{X_s^i : s \in T, s \leq t\}),\ t \in T,$$

where $\sigma(\{X(i,\cdot,s) : s \in T, s \leq t\})$ is the smallest σ-algebra containing all the P-null sets and with respect to $\mathcal{F}$ in which the random variables $\{X_s^i : s \in T, s \leq t\}$ are measurable.

Now, for $\omega \in \Omega$, let $\{\mathcal{G}_t^\omega\}_{t\in T}$ be the natural filtration generated by the empirical process X^ω, i.e.,

$$\mathcal{G}_t^\omega := \sigma(\{X_s^\omega : s \in T, s \leq t\}),\ t \in T,$$

where $X_s^\omega := X(\cdot,\omega,s)$. It is obvious that the empirical process X^ω is $\{\mathcal{G}_t^\omega\}$-adapted.

Note that X can be viewed as a stochastic process itself with the time parameter space T and the sample space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. It thus also generates a natural filtration on the Loeb product space, which is denoted by

$$\mathcal{H}_t := \sigma(\{X_s : s \in T, s \leq t\}), \quad t \in T,$$

where $X_s := X(\cdot, \cdot, s)$. $\{X_t\}_{t \in T}$ is $\{\mathcal{H}_t\}_{t \in T}$-adapted.

It is clear that martingales with respect to the above three natural filtrations can be defined as in the case of $\{\mathcal{F}_t^i\}_{t \in T}$.

Definition *(1) Two real-valued stochastic processes* φ *and* ψ *on the same sample space with time parameter space* T *are said to be independent, if, for any positive integers* m, n*, and for any* $t_1^1, \cdots, t_m^1$ *in* T*, and* $t_1^2, \cdots, t_n^2$ *in* T*, the random vectors* $(\varphi_{t_1^1}, \cdots, \varphi_{t_m^1})$ *and* $(\psi_{t_1^2}, \cdots, \psi_{t_n^2})$ *are independent.*

(2) We say that the stochastic processes $\{X^i, i \in I\}$ *are essentially independent, if, for* $\lambda \boxtimes \lambda$*-almost all* $(i_1, i_2) \in I \times I$*, the stochastic processes* X^{i_1} *and* X^{i_2} *are independent.*

(3) Two real-valued stochastic processes φ *and* ψ *on some (possibly different) sample spaces with time parameter space* T *are said to have the same finite dimensional distributions, if, for any* $t_1, \cdots, t_n \in T$*, the random vectors* $(\varphi_{t_1}, \cdots, \varphi_{t_n})$ *and* $(\psi_{t_1}, \cdots, \psi_{t_n})$ *have the same distribution.*

(4) We say that the stochastic processes $\{X^i, i \in I\}$ *have essentially the same finite dimensional distributions if there is a real-valued stochastic process* Y *with time parameter space* T *such that for* λ*-almost all* $i \in I$*, the stochastic processes* X^i *and* Y *have the same finite dimensional distributions.*

Note that the essential independence of the stochastic processes $\{X^i, i \in I\}$ as defined above only uses pairwise independence. Though pairwise independence and mutual independence are quite different for a countable collection of random variables (the first being, in general, weaker than the second), they are essentially equivalent for a continuum collection of random variables as well as of stochastic processes. We also note that if, for all $(i_1, i_2) \in I \times I$ with $i_1 \neq i_2$, X^{i_1} and X^{i_2} are independent, then the atomless property of λ implies that the stochastic processes $\{X^i, i \in I\}$ are essentially independent.

Following result says: (i) for a large collection of essentially independent martingales, the martingale property is preserved on the empirical processes essentially; (ii) a large collection of stochastic processes are martingales with respect to the natural filtration essentially iff so are the empirical processes

Theorem *(Albeverio-Sun-Wu, Trans. AMS)*
(1) Assume that the stochastic processes $\{X^i, i \in I\}$ *are essentially independent. If, for* λ*-almost all* $i \in I$*, the stochastic process* X^i *is an* $\{\mathcal{F}_t^i\}_{t\in T}$*-martingale on* $(\Omega, \mathcal{F}, P)$*, then, for* P*-almost all* $\omega \in \Omega$*, the empirical process* X^ω *is a* $\{\mathcal{G}_t^\omega\}_{t\in T}$*-martingale on* $(I, \mathcal{I}, \lambda)$*.*

(2)Assume that the stochastic processes $\{X^i, i \in I\}$ *are essentially independent and have essentially the same finite dimensional distributions. Then, the following are equivalent:*

- *(i) For* λ*-almost all* $i \in I$*, the stochastic process* X^i *is an* $\{\tilde{\mathcal{F}}_t^i\}_{t\in T}$*-martingale on* $(\Omega, \mathcal{F}, P)$*.*
- *(ii) For* P*-almost all* $\omega \in \Omega$*, the empirical process* X^ω *is a* $\{\mathcal{G}_t^\omega\}_{t\in T}$*-martingale on* $(I, \mathcal{I}, \lambda)$*.*

Let us present a spot of the proof to the above result. For this, we need *Law of Large Numbers*. Recall the usual law: If r. v.'s $X_j, j \in \mathbb{N}$, are iid with finite mean m, then the sum $\frac{1}{n}\sum_{j=1}^{n} X_j$ tends to the constant r.v. m almost surely.

Below is an exact law of large numbers for a continuum of essentially independent r.v.'s due to Y. Sun (PTRF, 1998)

Sun's Exact Law of Large Numbers   Let g be a process from the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a separable metric space S. If the random variables $g_i := f(i, \cdot)$ are essentially independent, i.e., for $\lambda \boxtimes \lambda$-almost all $(i_1, i_2) \in I \times I$, g_{i_1} and g_{i_2} are independent, then, for P-almost all $\omega \in \Omega$, the distribution μ_ω on S induced by the sample functions $g_\omega := f(\cdot, \omega)$ on I equals the distribution μ on S induced by g viewed as a random variable on $I \times \Omega$.

For the proof, let us only consider the simplest non-trivial case where $T = \{t_1, t_2\}$. Without loss of generality, we assume that $t_1 < t_2$. In this case, we only need to show

$$\mathbf{E}_\lambda\{X^\omega(t_2)|\mathcal{G}^\omega_{t_1}\} = X^\omega(t_1)\,, \quad \lambda - a.c.$$

for P-almost all $\omega \in \Omega$. While this is equivalent to

$$\mathbf{E}_\lambda\{[X^\omega(t_2) - X^\omega(t_1)]|\mathcal{G}^\omega_{t_1}\} = 0\,, \quad \lambda - a.c.$$

for P-almost all $\omega \in \Omega$. But this is true iff for any bounded Borel (measurable) function $h : \mathbb{R} \to \mathbb{R}$,

$$\int_I h(X(i,\omega,t_1))[X(i,\omega,t_2) - X(i,\omega,t_1)]d\lambda(i) = 0$$

for P-almost all $\omega \in \Omega$. Hence, it suffices to check this. Since X^i is an $\{\mathcal{F}^i_t\}_{t\in T}$-martingale for λ-almost all $i \in I$, we have

$$\mathbf{E}_P\{[X(i,\omega,t_2) - X(i,\omega,t_1)]|\mathcal{F}^i_{t_1}\} = 0 \quad P - a.c.$$

Thus, for any bounded Borel function $h : \mathbb{R} \to \mathbb{R}$,

$$\int_\Omega h(X(i,\omega,t_1))[X(i,\omega,t_2) - X(i,\omega,t_1)]dP(\omega) = 0$$

for λ-almost all $i \in I$.

Now by the essentially independence of $\{X^i, i \in I\}$, the family

$$\{h(X(i,\omega,t_1))[X(i,\omega,t_2) - X(i,\omega,t_1)], i \in I\}$$

is also essentially independent. Hence, by Sun's Exact Law of Large Numbers

$$\begin{aligned} & \int_I h(X(i,\omega,t_1))[X(i,\omega,t_2) - X(i,\omega,t_1)] d\lambda(i) \\ = & \int_{I\times\Omega} h(X(i,\omega,t_1))[X(i,\omega,t_2) - X(i,\omega,t_1)] d\lambda \boxtimes P(i,\omega) \end{aligned}$$

for P_L-almost all $\omega \in \Omega$. Moreover, by Keisler's Fubini theorem, we get

$$\begin{aligned} & \int_{I\times\Omega} h(X(i,\omega,t_1))[X(i,\omega,t_2) - X(i,\omega,t_1)] d\lambda \boxtimes P(i,\omega) \\ = & \int_I \left\{ \int_\Omega h(X(i,\omega,t_1)) \right. \\ & \qquad \left. \times [X(i,\omega,t_2) - X(i,\omega,t_1)] dP(\omega) \right\} d\lambda(i) \\ = & \ 0 \end{aligned}$$

and we are done.

Extensions

Straightforward extensions to the cases of submartingales and supermartingales.

Questions in future work: the cases of local martingales and more generally semimartingales.