ON THE STRUCTURE OF FIXED-POINT SETS OF NONEXPANSIVE MAPPINGS

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Banach's Contraction Principle. Let (M, ρ) be *a complete metric space and* $T : M \to M$ *a contraction:*

$$\rho (Tx, Ty) \leq k\rho (x, y)$$

for some $k < 1$ and *every* $x, y \in M$.

Then T *has a unique fixed point:* $Tx_0 = x_0$.

Definition. A mapping $T : M \to M$ is called nonexpansive if

$$\rho(Tx, Ty) \le \rho(x, y)$$

for every $x, y \in M$.

Our standard assumptions:

C - a bounded closed and convex subset of a Banach space X,

$T : C \to C$ - nonexpansive:

$$\|Tx - Ty\| \le \|x - y\|.$$

Example: $X = l^1$,

$$C = \left\{ (x_n) \in l^1 : x_n \ge 0, \|x\| = 1 \right\},$$

$$Tx = T(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

Then $T : C \to C$ is an isometry without fixed points.

Definition. We say that a Banach space X has the fixed point property (FPP) if every nonexpansive mapping $T : C \to C$ (defined on a closed convex bounded set C) has a fixed point: $Tx = x$.

The first existence results were obtained by F. Browder, D. Göhde and W. A. Kirk in 1965.

Problem:

- Does reflexivity imply FPP?
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Theorem (Maurey [1981], Dowling, Lennard [1997]). *Let $X = L_1[0,1]$ and Y be a (closed) subspace of X. Then*

$$Y \text{ is reflexive iff } Y \text{ has FPP.}$$

Let $T : C \to C$ be nonexpansive, fix $x_0 \in C$, and put

$$T_n x = \frac{1}{n} x_0 + \left(1 - \frac{1}{n}\right) Tx, \; x \in C.$$

Then $T_n : C \to C$ is a contraction and, by Banach's contraction principle, there exists $x_n \in C$ such that

$$T_n x_n = x_n.$$

Consequently, we obtain the so-called approximate fixed point sequence (x_n) for T:

$$\lim_{n \to \infty} \|Tx_n - x_n\| = 0.$$

Question:

- **Let $T, S : C \to C$ be commuting, nonexpansive mappings: $T \circ S = S \circ T$. Does there exist a sequence (x_n) such that**

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = 0?$$

Nonstandard reformulation:

Consider ${}^*T$, ${}^*S$: ${}^*C \to {}^*C$ and define nonexpansive mappings

$$\widehat{T}, \widehat{S} : \widehat{C} \to \widehat{C}$$

by putting

$$\widehat{T}\left({}^\circ \mathsf{x}\right) = {}^\circ\left({}^*T\mathsf{x}\right), \ \ \widehat{S}\left({}^\circ \mathsf{x}\right) = {}^\circ\left({}^*T\mathsf{x}\right),$$

where

$$\widehat{C} = {}^\circ\left({}^*C\right) = \{{}^\circ \mathsf{x} : \mathsf{x} \in {}^*C\}$$

and

$$°\mathsf{x} = \{\mathsf{y} \in {}^*E : \|\mathsf{x} - \mathsf{y}\|_* \approx 0\}$$

denotes the (generalized) standard part of x).

Question:

- Does there exist $\hat{x} \in \widehat{C}$ such that $\widehat{T}\hat{x} = \widehat{S}\hat{x} = \hat{x}$?

Definition. $Fix\, T$ is said to be a nonexpansive retract of C if there exists a nonexpansive mapping $r : C \to Fix\, T$ such that

$$rx = x$$

for every $x \in Fix\, T$.

Theorem ([2003]). Suppose $T, S : C \to C$ are commuting nonexpansive mappings and $Fix\,\widehat{T}$ is a nonexpansive retract of $\widehat{C}$. Then there exists $\hat{x} \in \widehat{C}$ such that

$$\widehat{T}\hat{x} = \widehat{S}\hat{x} = \hat{x}.$$

and, consequently,

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = 0$$

for some (x_n).

Proof sketch: the mixture of Bruck's ideas [1973] and (iterated) nonstandard techniques.

Let $r : \widehat{C} \to \mathrm{Fix}\,\widehat{T}$ be a nonexpansive retraction onto $\mathrm{Fix}\,\widehat{T}$.

- By transfer, $\widehat{r} : \widehat{\widehat{C}} \to \widehat{\mathrm{Fix}\, \widehat{T}}$ is a non-expansive retraction in the (double) nonstandard hull $\widehat{\widehat{X}}$.

- If $x \in \mathrm{Fix}\, \widehat{T}$, then $\widehat{T} \circ \widehat{S}\, x = \widehat{S} \circ \widehat{T}\, x = \widehat{S}\, x$ and hence $\widehat{S}\left(\mathrm{Fix}\, \widehat{T}\right) \subset \mathrm{Fix}\, \widehat{T}$. By transfer, $\widehat{\widehat{S}}\left(\widehat{\mathrm{Fix}\, \widehat{T}}\right) \subset \widehat{\mathrm{Fix}\, \widehat{T}}$.

- If $(\widehat{\widehat{S}} \circ \widehat{r})\, x = x$, then $x \in \widehat{\mathrm{Fix}\, \widehat{T}}$, $\widehat{r}x = x$, (since $\widehat{r}$ is a retraction), and consequently $(\widehat{\widehat{S}} \circ \widehat{r})\, x = \widehat{\widehat{S}}x = x$. (Bruck's argument).

- Hence

$$\widehat{\mathrm{Fix}\, \widehat{T}} \cap \mathrm{Fix}\, \widehat{\widehat{S}} = \mathrm{Fix}\, (\widehat{\widehat{S}} \circ \widehat{r}) \neq \emptyset,$$

(it follows from $\aleph_1$-saturation and the existence of an approximate fixed point sequence:

$$\widehat{\widehat{S}} \circ \widehat{r} : \widehat{\widehat{C}} \to \widehat{\mathrm{Fix}\,\widehat{T}}$$

is a nonexpansive and neocontinuous mapping defined on a neocompact set $\widehat{\widehat{C}}$).

- But

$$\mathrm{Fix}\,\widehat{\widehat{T}} \cap \mathrm{Fix}\,\widehat{\widehat{S}} \supset \widehat{\mathrm{Fix}\,\widehat{T}} \cap \mathrm{Fix}\,\widehat{\widehat{S}} \neq \emptyset$$

and consequently

$$\lim_{n\to\infty} \left\| \widehat{T}\,x_n - x_n \right\| = \lim_{n\to\infty} \left\| \widehat{S}\,x_n - x_n \right\| = 0.$$

for some sequence (x_n) in $\widehat{C}$.

- By neocompactness again,

$$\mathrm{Fix}\,\widehat{T} \cap \mathrm{Fix}\,\widehat{S} \neq \emptyset.$$

Question:

- **If $T : C \to C$ is a nonexpansive mapping, is then $Fix\,\widehat{T}$ a nonexpansive retract of $\widehat{C}$?**

(Note that $Fix\,T$ need not be a nonexpansive retract of C but a mapping $\widehat{T} : \widehat{C} \to \widehat{C}$ is much more regular).

Theorem ([2006]). ***For any (at most) countable set $A \subset \mathrm{Fix}\,\widehat{T}$ there exists a nonexpansive mapping $r : \widehat{C} \to \mathrm{Fix}\,\widehat{T}$ such that $rx = x$ for $x \in A$.***

Proof sketch:

- Fix $\mathsf{x} \in {}^*C$, $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$, and consider an (internal) mapping $\mathsf{T}_\mathsf{x} : {}^*C \to {}^*C$ defined by
$$\mathsf{T}_\mathsf{x}\mathsf{z} = \frac{1}{\omega}\mathsf{x} + \left(1 - \frac{1}{\omega}\right) {}^*T\mathsf{z}, \ \mathsf{z} \in {}^*C.$$
- By transfer of the Banach Contraction Principle, there exists exactly one point, say, $\mathsf{F}_\omega\mathsf{x} \in {}^*C$ such that $\mathsf{T}_\mathsf{x}\mathsf{F}_\omega\mathsf{x} = \mathsf{F}_\omega\mathsf{x}$. This defines a mapping $\mathsf{F}_\omega : {}^*C \to {}^*C$ which is *-nonexpansive. Moreover
$$\mathsf{T}_\mathsf{x}\mathsf{F}_\omega\mathsf{x} = \mathsf{F}_\omega\mathsf{x} = \frac{1}{\omega}\mathsf{x} + \left(1 - \frac{1}{\omega}\right) {}^*T\mathsf{F}_\omega\mathsf{x}$$
for $\mathsf{x} \in {}^*C$.
- Hence
$$\left\| {}^*T\mathsf{F}_\omega\mathsf{x} - \mathsf{F}_\omega\mathsf{x} \right\|_* \le \frac{1}{\omega}\mathrm{diam}C$$
and
$$\left\| \mathsf{F}_\omega\mathsf{x} - \mathsf{x} \right\|_* \le (\omega - 1) \left\| {}^*T\mathsf{x} - \mathsf{x} \right\|_* .$$

- Put

$$r_\omega {}^\circ\mathsf{x} = {}^\circ\left(\mathsf{F}_\omega \mathsf{x}\right), \ \mathsf{x} \in {}^*C.$$

and notice that $r_\omega : \widehat{C} \to \mathrm{Fix}\widehat{T}$ is a well-defined nonexpansive mapping.

- By $\aleph_1$-saturation, for any countable set $A \subset \mathrm{Fix}\,\widehat{T}$ there exists $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that

$$rx = x \quad \text{for} \quad x \in A.$$

(The argument is not very easy in the language of Banach space ultraproducts).

For more details:

A. Wiśnicki, *On fixed-point sets of nonexpansive mappings in nonstandard hulls and Banach space ultrapowers*, Nonlinear Anal., to appear.