Scaling in a metric space

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The main part of my talk is a personal development starting from a joint work in progress with K. Tchizawa, R. Nishiyama and N. Kakiuchi.

A better title for this talk would be:

A new kind of space for scaling

Actually, I want to show that, in the context of scaling, a new structure of space appears which is more general and perhaps more accurate than the structure of metric space.

1. Scaling in Scientific Field

The concept of scale is fundamental in any empirical science.

Moreover, any change of scale or scaling deeply affects the objects which are studied by this kind of science.

For instance, this is the case in Image Processing and also in Geographical Information Systems and Spatial Analysis.

Topology is a major element of these two fields. However, it is noted that a scaling can cause deformations of the apparent topology.

Considered at some scale, a city A *can be inside a geographical area* B.
Considered at a smaller scale, the same city A *may be on the boundary of* B.

The scientists working in these fields must take into account these topological deformation phenomena. It is a real problem to identify a given object represented at different scales.

The mathematical transformation naturally related to a scaling is the notion of homothety (for instance in an affine space).

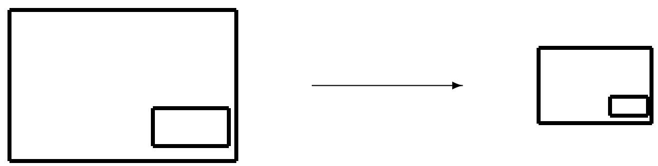
But *an homothety is an homeomorphism; thus, it leaves invariant the topology.*

On the basis of this observation, many experts in Geographical Information Systems and Spatial Analysis concluded that a scaling is a natural transformation which cannot be exactly represented by a mathematical transformation.

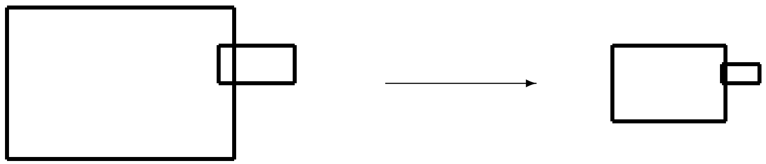
In some works,this limitation was circumvented by giving rules which describe what are the topological effects of a scaling.

This rules have the role of axioms which replace a mathematical definition of a scaling.

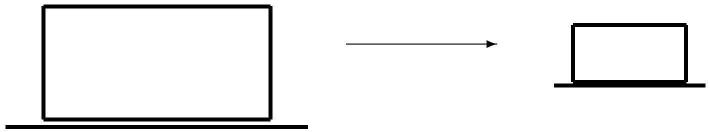
This is the case in the work of T.Y. Jen (1999); he stated 3 main constraints (C_1), (C_2) and (C_3) for a scaling.



(C_1) If $A \subset B$ at a large scale, then $A \subset B$ at a small scale.

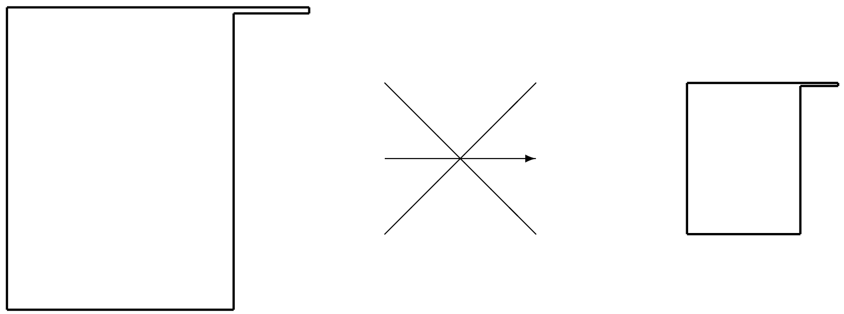


(C_2) If $A \cap B \neq \emptyset$ at a large scale, then $A \cap B \neq \emptyset$ at a small scale.



(C_3) If $A \cap (B)^o = \emptyset$ at a large scale, then $A \cap (B)^o = \emptyset$ at a small scale.

These rules are valid under a condition of topological invariance only explained by the following figure:



Unfortunately, these axioms are not clear as long as we do not have a mathematical definition of a scaling.

2. A Nonstandard Approach

The preceding point of view according to which scalings are not representable in mathematics was not correct because it was based on a incomplete analysis of the notion of (empirical) scaling. In fact, we can agree that a scaling is the union of two distinct but dependant processes:

(1) an *homothety* which *strongly* changes the size of any object;

(2) a *simplification* which allows to *neglect* too small details.

In order to build a convenient mathematical concept of scaling, we have to translate simultaneously these two processes.

Nonstandard analysis seems to be a good tool likely to bring a suitable answer to this question.

Two reasons:

- Nonstandard analysis extends the field of real numbers by introducing infinitely large and small numbers. With this new numbers, it is possible to define a notion of strong homothety.

- Nonstandard analysis gives a precise meaning to the relation "x is infinitely near y" with which it is possible to specify a process of simplification of too small details.

Let (X,d) be a metric space. We consider a superstructure $V(S)$ over a set S such that $(X \cup \mathbb{R}) \subset S$ and a nonstandard model of $V(S)$

$$\begin{array}{ccc} V(S) & \longrightarrow & V({}^*S) \\ X & \longmapsto & {}^*X \end{array}$$

of $V(S)$ with a large enough saturation property (for instance we may choose a polysaturated model). Within this framework, we can used the set ${}^*\mathbb{R}$ of hyperreal numbers and the standard part map $\mathrm{st} : {}^*\mathbb{R} \to \mathbb{R} \cup \{+\infty, -\infty\}$

$$\forall u \in {}^*\mathbb{R} \quad \mathrm{st}(u) = \begin{cases} {}^o u \in \mathbb{R} \text{ such that } {}^o u \simeq u & \text{if } u \text{ is limited} \\ +\infty & \text{if } u \simeq +\infty \\ -\infty & \text{if } u \simeq -\infty \end{cases}$$

Each element of ${}^*\mathbb{R}^*_+ := \{\gamma \in {}^*\mathbb{R} \ ; \ 0 < \gamma\}$ is called a *scale*.

Given a scale $\alpha \in {}^*\mathbb{R}^*_+$, we define the equivalence relation $\simeq_\alpha$ on ${}^*X$ defined by

$$\forall (x, y) \in {}^*X^2 \ \ (x \simeq_\alpha y \iff \alpha \, {}^*d(x, y) \simeq 0)$$

Then, we introduce the quotient set $X_\alpha = {}^*X / \simeq_\alpha$ and the canonical projection

$$\begin{array}{rcccc} \pi_\alpha & : & {}^*X & \longrightarrow & X_\alpha \\ & & x & \longmapsto & \pi_\alpha(x) \end{array}$$

where $\pi_\alpha(x)$ denotes the equivalence class of $x \in {}^*\mathbb{R}$, i.e the set of $y \in {}^*X$ such that $x \simeq_\alpha y$. For every $(x, y) \in {}^*X^2$, let $\delta_\alpha(\pi_\alpha(x), \pi_\alpha(y)) = \mathrm{st}(\alpha \, {}^*d(x, y)) \in \mathbb{R}^*_+ \cup \{+\infty\}$.

The construction of X_α and δ_α implements the two fundamental aspects (strong homothety and simplification of the details) of an empirical scaling.

It seems to me that it could be interesting to find a richer structure which would carry all the information contained in a scaling.

In fact, I think now that the good objects on which scalings are naturally defined are not metric spaces but suitable generalizations of them.

3. A generalization of the notion of metric space

A *generalized distance* on a set F is a map

$$\delta_\alpha : F \times F \to \mathbb{R}_+ \cup \{+\infty\}$$

such that, for all $x, y, z \in F$:

1. $\delta(x, x) = 0$
2. $\delta(x, y) = \delta(y, x) > 0$ for $x \neq y$
3. $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$

If δ is a generalized distance on F, we can define the open balls:

$$\forall (x, r) \in F \times (\mathbb{R}_+^* \cup \{+\infty\}) \quad B_\delta(x, r) = \{y \in F \ ; \ \delta(x, y) < r\}$$

The family of open balls is clearly a basis of a topology on F.

The open balls of radius $+\infty$ are also the equivalence class for the relation $\delta(x, y) < +\infty$. These large open balls are obviously metric spaces for δ and are called the *metric components of F for δ*. Let $\mathcal{M}_F$ be the set of all metric components of F for δ and, for each $x \in F$, let $C_F(x)$ the element $E \in \mathcal{M}_F$ such that $x \in E$.

Consequently, a set provided with a generalized distance is just the disjoint union of a family of metric spaces. We want to improve this concept by the consideration of a kind of distance on the set $\mathcal{M}_F$ of metric components of F. For that, we need an external group quotient of ${}^*\mathbb{R}$.

The galaxy of 0 is the external set $\mathbb{G}$ of limited numbers

$$\mathbb{G} = \{t \in {}^*\mathbb{R} \ ; \ \exists n \in \mathbb{N} \ |t| \leq n\}$$

This is an external additive subgroup of ${}^*\mathbb{R}$.

Now, we introduce the quotient group $\mathbb{G}({}^*\mathbb{R}) = {}^*\mathbb{R}/\mathbb{G}$ and the canonical projection

$$\begin{array}{lllll} \mathsf{Gal} & : & {}^*\mathbb{R} & \longrightarrow & \mathbb{G}({}^*\mathbb{R}) \\ & & t & \longmapsto & \mathsf{Gal}(t) \end{array}$$

where Gal(t) denotes the equivalence class of $t \in {}^*\mathbb{R}$, i.e the set of $s \in {}^*\mathbb{R}$ such that $s - t \in \mathbb{G}$ which is also the galaxy $t + \mathbb{G}$ of t.

There is a natural relation $\leq$ on $\mathbb{G}({}^*\mathbb{R})$ defined by

$$\forall (s,t) \in {}^*\mathbb{R}^2 \ \ [\mathsf{Gal}(s) \leq \mathsf{Gal}(t) \iff (s \leq t \text{ or } s - t \in \mathbb{G})]$$

We see at once that $\leq$ is a total order relation. Moreover, this relation is compatible with the additive structure of $\mathbb{G}({}^*\mathbb{R})$

$$(0 \leq \mathsf{Gal}(s) \text{ and } 0 \leq \mathsf{Gal}(t)) \implies 0 \leq \mathsf{Gal}(s) + \mathsf{Gal}(t)$$

These properties means that $\mathbb{G}({}^*\mathbb{R})$ is an ordered additive group and Gal is a not decreasing group morphism.

A *galactic distance* (we say a $\mathbb{G}$-*distance*) on a set $\mathcal{E}$ is a map $\Delta : \mathcal{E} \times \mathcal{E} \to \mathbb{G}({}^*\mathbb{R})_+$ such that, for every $E_1, E_2, E_3 \in \mathcal{E}$

1. $\Delta(E_1, E_1) = 0$

2. $\Delta(E_1, E_2) = \Delta(E_2, E_1) > 0$ for $E_1 \neq E_2$

3. $\Delta(E_1, E_3) \leq \Delta(E_1, E_2) + \Delta(E_2, E_3)$

If Δ is a $\mathbb{G}$-distance on a set $\mathcal{E}$, there is a well defined topology on $\mathcal{E}$ so that the family of open balls is a basis of this topology.

Définition 1. *A* galactic space *(we say a* $\mathbb{G}$-space *) is a structure* (F, δ, Δ) *such that:* F *is a set,* δ *is a generalized distance on* F *and* Δ *is a* $\mathbb{G}$*-distance on the set* $\mathcal{M}_F$ *of metric components of* F *for* δ*.*

If (F, δ, Δ) is a $\mathbb{G}$-space, then δ is a true distance on each metric component of F for δ and Δ is a kind of distance on the set $\mathcal{M}_F$ of metric components of F for δ. In other words, a $\mathbb{G}$-space is a set $\mathcal{M}_F$ of metric spaces with moreover a topology on $\mathcal{M}_F$ induced by a $\mathbb{G}$-distance.

Let $\gamma \in {}^*\mathbb{R}_+$ be a limited number and $\tau \in \mathbb{G}({}^*\mathbb{R})$. There exists $t \in {}^*\mathbb{R}$ such that $\tau = \text{Gal}(t)$ and we define the external set $\gamma.\tau = \{\gamma s \ ; \ s \in \tau\}$ so that $\gamma.\tau = \gamma t + \gamma\mathbb{G}$.

Thus, $\gamma.\tau = \text{Gal}(\gamma t)$ if γ is appreciable and $\gamma.\tau \subset \text{Hal}(\gamma t) \subset \text{Gal}(\gamma t)$ if $\gamma \simeq 0$. Then, we introduce the element $\gamma \bullet \tau$ of $\mathbb{G}({}^*\mathbb{R})$ such that $\gamma \bullet \tau = \text{Gal}(\gamma t)$; hence $\gamma.\tau \subset \gamma \bullet \tau$.

Moreover, if $\tau \in \mathbb{G}({}^*\mathbb{R})^*_+$, the standard part map is constant on the set $\gamma.\tau$ and this constant value is named $\text{st}(\gamma.\tau)$. Indeed, such a τ can be written $\text{Gal}(t)$ for some $t \in {}^*\mathbb{R}_+$ such that $t \simeq +\infty$; therefore every element of $\gamma.\tau$ is infinitely large if $\gamma \not\simeq 0$ and $\gamma.\tau \subset \text{Hal}(\gamma t)$ if $\gamma \simeq 0$.

Given $\gamma \in {}^*\mathbb{R}^*_+$ such that $0 < \gamma \leq 1$, a *γ-contraction* of (F, δ, Δ) is a $\mathbb{G}$-space (F', δ', Δ') and a surjective map $f : F \to F'$ such that:

(1) $\forall (x_1, x_2) \in F^2$

$$\delta'(f(x_1), f(x_2)) = \begin{cases} \text{st}(\gamma\, \delta(x_1, x_2)) & \text{if } \delta(x_1, x_2) < +\infty \\ \text{st}(\gamma . \Delta(E_1, E_2)) & \text{if } \delta(x_1, x_2) = +\infty \end{cases}$$

where $E_1 := C_F(x_1)$ and $E_2 := C_F(x_2)$

(2) $\forall (E_1, E_2) \in \mathcal{M}_F^2$

$$\Delta'(f(E_1), f(E_2)) = \gamma \bullet \Delta(E_1, E_2)$$

A γ-contraction of a $\mathbb{G}$-space is a kind of quotient of this space.

Given a $\mathbb{G}$-space (F, δ, Δ) and γ a number in ${}^*\mathbb{R}^*_+$ such that $0 < \gamma \leq 1$, we can see that there exists a γ-contraction of (F, δ, Δ).

Moreover, two γ-contractions of the same $\mathbb{G}$-space (F, δ, Δ) are essentially the same, i.e. are isomorphic in a natural way.

Remark Actually, the concept of galactic space is more general. We can define the same kind of structure in the algebraic context of a proper ordered field extension $i : \mathbb{R} \to \mathbb{K}$. All what we need: the two subgroups
$\mathrm{Hal}(0) = \{x \in \mathbb{K} \ ; \ \forall n \in \mathbb{N}^* \ |x| \leq 1/n\}$
$\mathrm{Gal}(0) = \{x \in \mathbb{K} \ ; \ \exists n \in \mathbb{N} \ |x| \leq n\}$
In this framework, $\mathrm{Gal}(0)/\mathrm{Hal}(0) \approx \mathbb{R}$ and then, there exists a "standard part" map st : $\mathbb{K} \to \mathbb{R} \cup \{+\infty, -\infty\}$ with the usual properties. Moreover, the quotient group $\mathbb{G}(\mathbb{K}) := \mathbb{K}/\mathrm{Gal}(0)$ is an ordered group. Then, we can define a notion of galactic distance, of galactic space and of γ-contraction (for $\gamma \in \mathbb{K}$ such that $0 < \gamma \leq 1$).

For instance, let us consider the field $\mathbb{L}$ of Laurent formal series $f = \sum_{i=m}^{+\infty} a_i X^i$ such that $m \in \mathbb{Z}$ and $a_i \in \mathbb{R}$ for each i. There is a well defined structure of ordered field extension of $\mathbb{R}$ on $\mathbb{L}$ such that $X > 0$ and $X \simeq 0$. Then, $\mathsf{Hal}(0) = X\mathbb{R}[[X]]$, $\mathsf{Gal}(0) = \mathbb{R}[[X]]$ and $\mathbb{L}/\mathsf{Gal}(0) \equiv X^{-1}\mathbb{R}[X^{-1}]$.

Hence, on the set $\mathbb{R}[X^{-1}]$ we introduce the generalized distance

$$\delta(f,g) = \begin{cases} |f-g| & \text{if } f-g \in \mathbb{R} \\ +\infty & \text{else} \end{cases}$$

The metric set of metric components is

$$\mathcal{M} = \{\mathbb{R} + h \ ; \ h \in X^{-1}\mathbb{R}[X^{-1}]\}$$

Then we define a galactic distance

$$\begin{array}{rcll} \Delta & : & \mathcal{M} \times \mathcal{M} & \longrightarrow & X^{-1}\mathbb{R}[X^{-1}] \\ & & (\mathbb{R}+h, \mathbb{R}+h') & \longmapsto & |h-h'| \end{array}$$

Thus, for every $f \in X\mathbb{R}[[X]]$, there is a f-contraction of the galactic space $(\mathbb{R}[X^{-1}], \delta, \Delta)$. For instance, we can verify that the X-contraction of this space is isomorphic to it.

4. Definition and first properties of a scaling

We consider again the metric space (X, d), a scale $\alpha \in {}^*\mathbb{R}_+^*$, the quotient set $X_\alpha = {}^*X / \simeq_\alpha$ for the relation $\alpha\, {}^*d(x, y) \simeq 0$, the canonical projection π_α and, for every $(x, y) \in {}^*X^2$, the element $\delta_\alpha(\pi_\alpha(x), \pi_\alpha(y)) = \mathrm{st}(\alpha\, {}^*d(x, y)) \in \mathbb{R}_+^* \cup \{+\infty\}$.

It is clear that δ_α is a generalized distance on X_α. Let $\mathcal{M}_\alpha$ be the set of metric components of X_α for δ_α. Actually each $E \in \mathcal{M}_\alpha$ is of the following form

$$E = \mathrm{Cone}(X, x_E, \alpha) := \{x \in {}^*X \ ; \ \alpha\, {}^*d(x, x_E) \not\simeq +\infty\} / \simeq_\alpha$$

where x_E is any point in ${}^*X$ such that $\pi_\alpha(x_E) \in E$. When $\alpha \simeq 0$, the set $\mathrm{Cone}(X, x_E, \alpha)$ is exactly the so-called *asymptotic cone* of (X, d) with respect to x_e and α.

We recall that Gal is the canonical projection of ${}^*\mathbb{R}$ onto the group $\mathbb{G}({}^*\mathbb{R}) = {}^*\mathbb{R}/\mathrm{Gal}(0)$ of its galaxies.

For every E_1 and E_2 in $\mathcal{M}_\alpha$, we choose x_{E_1} and x_{E_2} in ${}^*X$ such that $E_1 = \mathrm{Cone}(X, x_{E_1}, \alpha)$ and $E_2 = \mathrm{Cone}(X, x_{E_2}, \alpha)$; then we define

$$\Delta_\alpha(E_1, E_2) = \mathrm{Gal}(\alpha\, {}^*d(x_{E_1}, x_{E_2}))$$

We get a well defined map $\Delta_\alpha : \mathcal{M}_\alpha^2 \to \mathbb{G}({}^*\mathbb{R})$ which is a $\mathbb{G}$-distance.

Then, the *α-scaling* of the metric space (X, d) is the $\mathbb{G}$-space $(X_\alpha, \delta_\alpha, \Delta_\alpha)$.

We consider now two scales $\alpha, \beta \in {}^*\mathbb{R}^*_+$ such that $\beta \leq \alpha$.

The main relation between the corresponding scalings of (X, d) is the following.

Theorem 1. *There is a natural map* $\pi_{\beta,\alpha} : X_\alpha \to X_\beta$ *such that* $\pi_\beta = \pi_{\beta,\alpha} \circ \pi_\alpha$. *Moreover,this map defines a* β/α*-contraction of the* α*-scaling* $(X_\alpha, \delta_\alpha, \Delta_\alpha)$ *of* (X, d) *to its* β*-scaling* $(X_\beta, \delta_\beta, \Delta_\beta)$

Consequently, insofar as we are only concerned by the structure of $\mathbb{G}$-space, we can define the β-scaling of (X, d) using only its α-scaling. It seems that this property is not true for the asymptotic cones.

If $(\beta/\alpha) \not\simeq 0$ then, the α-scaling $(X_\alpha, \delta_\alpha, \Delta_\alpha)$ and the β-scaling $(X_\beta, \delta_\beta, \Delta_\beta)$ of (X, d) are such that $X_\beta = X_\alpha$, $\delta_\beta = \mathrm{st}(\beta/\alpha)\,\delta_\alpha$ and $\Delta_\beta = (\beta/\alpha) \bullet \Delta_\alpha$.

If $\beta/\alpha \simeq 0$, then, for every $\xi, \eta \in X_\alpha$, if $E, F \in \mathcal{M}_\alpha$ are such that $\xi \in E$ and $\eta \in F$, then

1. $\pi_{\beta,\alpha}(\xi) = \pi_{\beta,\alpha}(\eta) \iff (\beta/\alpha).\Delta_\alpha(E, F) \subset \mathrm{Hal}(0)$

2. $\delta_\beta(\pi_{\beta,\alpha}(\xi), \pi_{\beta,\alpha}(\eta)) = \mathrm{st}((\beta/\alpha).\Delta_\alpha(E, F))$

3. $\Delta_\beta(\pi_{\beta,\alpha}(E), \pi_{\beta,\alpha}(F) = (\beta/\alpha) \bullet \Delta_\alpha(E, F)$

The main topological relations between the two scalings are the following ones.

Proposition 1. *We suppose that* $\alpha, \beta \in {}^*\mathbb{R}^*_+$ *are such that* $\beta \leq \alpha$ *and we consider the maps* $\pi_{\beta,\alpha} : X_\alpha \rightarrow X_\beta$ *and* $\pi_{\beta,\alpha} : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta$*.*
(1) This maps are continuous.
(2) If $\beta/\alpha \not\simeq 0$*, this maps are homeomorphisms.*
(3) If $\beta/\alpha \simeq 0$*, then for every* $(\xi, E') \in X_\beta \times \mathcal{M}_\beta$*,* $(\pi_{\beta,\alpha})^{-1}(\{\xi'\})$ *is an open set of* X_α *and* $(\pi_{\beta,\alpha})^{-1}(E')$ *is an open set of* $\mathcal{M}_\beta$*.*

Within our definition of a scaling, it is possible (and easy) to give a precise statement and a proof for each of the 3 constraints of Jen.

More generally, an interesting problem is to find some general rules for the topological deformations induced by our notion of scaling.