Abstract. Many interesting results in topology and functional analysis are closely related to situations in which two otherwise distinct topologies or uniformities coincide. In this talk, we consider a number of pairs of infinitesimal relations and examine the consequences of the condition that they coincide on certain subsets of the underlying space. One example leads to a new characterization of uniform spaces with invariant nonstandard hulls. Other applications include external characterization of strong and weak compactness in Banach spaces.

What happens when two infinitesimal Relations Coincide?

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Notation

- $(X, \mathcal{U})$: uniform space; $({}^*X, \simeq)$.
- (Z, d): a metric space; $({}^*Z, \simeq)$.
- $\mathcal{F}(X, Z) = \{f : f : X \to Z\}$.
- For each $\mathcal{V} \subseteq {}^*\mathcal{F}(X, Z)$, we define an infinitesimal relation $\simeq_v$ on ${}^*X$ by

$$a \simeq_v b \quad \leftrightarrow \quad f(a) \simeq f(b) \;\; \forall f \in \mathcal{V}.$$

and, when $f(x) \in \text{ns}({}^*Z)$ for each $x \in X$, we may define an infinitesimal relation $\approx_v$ on ${}^*X$ by

$$a \approx_v b \quad \leftrightarrow \quad {}^{*\circ}f(a) \simeq {}^{*\circ}f(b) \;\; \forall f \in \mathcal{V}.$$

Problem

Investigate the consequences of conditions such as:

- $\simeq\, = \,\simeq_v$ or $\simeq\, = \,\approx_v$ on a subset of ${}^*X$.

And, assuming that $\mathcal{W} \subseteq \mathcal{V}$, investigate the condition

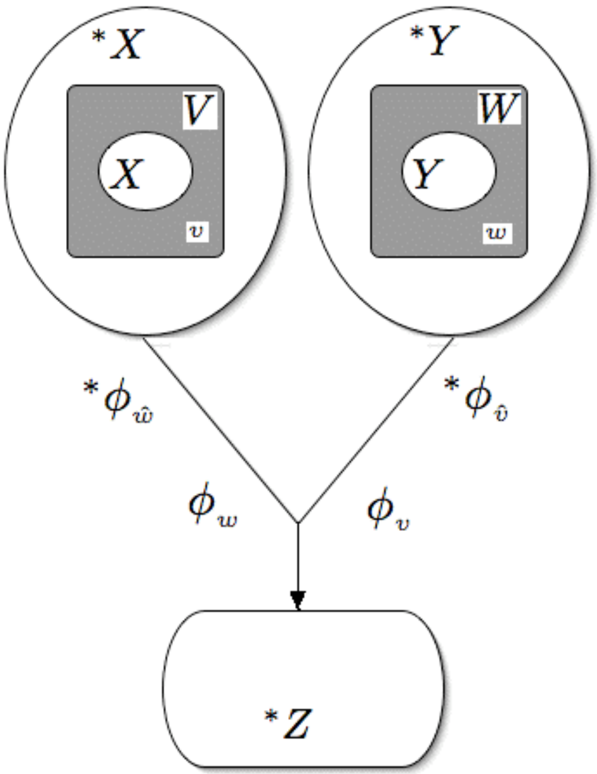
- $\simeq_v\, = \,\simeq_w$ on a subset of ${}^*X$, or
- $\approx_v\, = \,\approx_w$ on a subset of ${}^*X$.

The most fruitful cases are when $\mathcal{V}$ is a union monad and $\mathcal{W} = {}^\sigma\mathcal{V}$.

Example 1.

- X, Y: infinite sets; (Z, d) a metric space.
- $\phi : X \times Y \to Z$
- For each $w \in {}^*Y$, we have an internal function $\phi_w : {}^*X \to {}^*Z$; $\phi_w(v) = {}^*\phi(v, w)$.
- In case ${}^*\phi(x, w)$ is near-standard, for each $x \in X$, we have a standard function ${}^*\phi_{\hat{w}} : {}^*X \to {}^*Z$; $\phi_{\hat{w}}(x) = {}^\circ\phi(x, w)$.
- Similarly, for each $v \in {}^*X$, we define the functions: ${}^*\phi_{\hat{v}} : {}^*Y \to {}^*Z$ and $\phi_v : {}^*Y \to {}^*Z$.

$$^*\phi : \ ^*X \times \ ^*Y \to \ ^*Z$$



For each $a, b \in \ ^*X$, we define

$$a \ \simeq_W b$$

by

$$\phi_w(a) \simeq \phi_w(b) \text{ for all } w \in W$$

and

$$a \ \approx_W b$$

by

$$^*\phi_{\widehat{w}}(a) \simeq \ ^*\phi_{\widehat{w}}(b) \quad \text{for all } w \in W$$

Problem

Given that V and W are union monads with $X \subseteq V$ and $Y \subseteq W$, investigate the the consequences of conditions:

1. $\simeq_Y = \simeq_W$ on V and $\simeq_X = \simeq_V$ on W.
2. $\approx_Y = \approx_W$ on V and $\approx_X = \approx_V$ on W.

Some of our results concerning (1) have been published. Our results concerning (2) were presented at the 2004 conference in Aveiro, Portugal.

Example 2.

- Λ is a set of pseudometrics on X.
- For each $\rho \in \Lambda$, $p \in X$, we have a function $\rho_p : X \to \mathbb{R}$ given by $\rho_p(x) = \rho(x, p)$
- Consider two infinitesimal relations on ${}^*X$:

1.

$$a \simeq b \quad \leftrightarrow \quad {}^*\rho(a, b) \simeq 0; \qquad (\rho \in \Lambda).$$

2.

$$a \approx b \quad \leftrightarrow \quad {}^*\rho_p(a) \approx {}^*\rho_p(b); \quad (\rho \in \Lambda, p \in X).$$

In general, we have $\simeq \subseteq \approx$ on ${}^*X$.

Theorem. $\simeq = \approx$ on pns(${}^*X$).

Definition

Let fin(${}^*X$) denote the set of all $x \in {}^*X$ such that ${}^*\rho(x,p)$ is limited for each $\rho \in \Lambda$ and each $p \in X$. We call the uniform space (X, Λ) an S-space if

$$\simeq = \approx \qquad \text{on fin}({}^*X).$$

Theorem. Every compact space is an S-space.

Proof. We have $\simeq = \approx$ on $\mathrm{pns}({}^*X)$, and, in a compact space, we have

$$\mathrm{ns}({}^*X) = \mathrm{pns}({}^*X) = \mathrm{fin}({}^*X) = {}^*X.$$

Alternatively,

The uniform structure compatible with the topology of a compact space is unique. Hence we must have $\simeq = \approx$ on the entire ${}^*X$.

This leads us to the following criterion.

Notation: Let $C^b(X)$ denote the set of all bounded continuous functions on the uniform Hausdorff space (X, Λ) equipped with the topology of uniform convergence on X. Let $A(X)$ denote the subalgebra of $C(X)$ consisting of those $f \in C(X)$ that are constant on the complement of some compact set in X.

Theorem. The uniform space (X, Λ) is an S-space if $A(X)$ is dense in $C^b(X)$.

Proof. This condition is equivalent to the uniqueness of compatible uniform structures, and is due to I. S. GÁL, (1958).

Theorem. Every locally compact space equipped with the uniformity $\mathcal{U}_{\simeq}$ it inherits from its one-point compactification is an S-space.

Proof. It is well known that $\mathcal{U}_{\simeq}$ is the coarsest uniformity that is compatible with the topology of X (Alice Dickson, 1952). Since, in general, $\mathcal{U}_{\simeq}$ is finer than $\mathcal{U}_{\approx}$, it follows that $\mathcal{U}_{\approx} = \mathcal{U}_{\simeq}$

Theorem. Every pre-compact space is an S-space.

Proof. We have $\simeq\, =\, \approx$ on $\text{pns}({}^*X)$, and, in a pre-compact space, we have

$$\text{pns}({}^*X) = \text{fin}({}^*X) = {}^*X.$$

Theorem. Every uniform space with invariant nonstandard hulls is an S-space.

Proof. We have $\simeq = \approx$ on $\mathrm{pns}({}^*X)$, and, in a uniform space with invariant nonstandard hulls, we have

$$\mathrm{pns}({}^*X) = \mathrm{fin}({}^*X).$$

Theorem. *A uniform space (X, Λ) is an S-space if and only if it has invariant nonstandard hulls.*

Proof. Fix $p \in \text{fin}(\,^*X)$, $\rho \in \Lambda$, and $\epsilon \in \mathbb{R}^+$. Let

$$\mathcal{F} = \{B \in \mathcal{P}(X) : p \in \,^*B\},$$

and let

$$\mathcal{G} = \{B \in \mathcal{F} : \; \mathcal{U}_{\simeq} \; = \mathcal{U}_{\approx} \; \text{ on } B \; \}.$$

Let $\{F_1, \ldots, F_n\}$ be a *–finite subset of $^*\mathcal{F}$ that contains $\mathcal{F}$. Let $G = \cap_{i=1}^{n} F_i$. Then we have

$$\emptyset \neq G \subseteq \mu(\mathcal{F}) \subseteq \text{fin}(\,^*X).$$

Hence $G \in \,^*\mathcal{G}$, and $\mathcal{G} \neq \emptyset$. Pick a set $B \in \mathcal{G}$.

There exist $\delta \in \mathbb{R}^+$ and $p_1, \ldots, p_n \in X$ such that the set

$$U = \{\langle u, v\rangle \in B^2 : \max_i |\rho(u, p_i) - \rho(v, p_i)| < \delta\}.$$

is contained in the set

$$V = \{\langle u, v\rangle \in B^2 : \rho(u, v) < \epsilon\}.$$

Therefore, ${}^*U[x] \subseteq {}^*V[x]$. Now let $a_i = {}^\circ\rho(x, p_i)$, then $a_i \in \mathbb{R}$. Let $A = \{v \in B : \max_i |a_i - \rho(v, p_i)| < \frac{\delta}{2}\}$. Clearly, $A \subseteq X$ and $x \in {}^*A$. From the latter, it follows that $A \neq \emptyset$. Pick a point $q \in A$. Since ${}^*A \subseteq {}^*U[x] \subseteq {}^*V[x]$, we have ${}^*\rho(x, q) < \epsilon$. Hence $x \in \text{pns}({}^*X)$, and the proof is finished.