NONSTANDARD MATHEMATICS

Pisa, Italy

May 2006

ERNA + TRANSFER

Sam Sanders and Chris Impens

University of Ghent, Belgium

(thanks to Ulrich Kohlenbach,

T.U. Darmstadt)

Basic papers

R. Chuaqui and P. Suppes,

Free-variable axiomatic foundations of infinitesimal analysis: a fragment with finitary consistency proof (1995)

'A constructive system of NSA, meant to provide a foundation close to mathematical practice characteristic of theoretical physics.'

R. Sommer and P. Suppes,

Finite Models of Elementary Recursive Nonstandard Analysis (1996a)

Dispensing with the Continuum (1996b)

'Simpler + more versatile in allowing definition by recursion.'

ERNA =

Elementary Recursive Nonstandard Analysis

'By trading in the completeness axioms for axioms asserting the existence of infinitesimals, we end up with a system that is actually more constructive, and in many ways better matches certain geometric intuitions about the number line. (...) Many classical theorems that are used in mathematical practice have versions provable in ERNA.'

Introduction

Consistency of ERNA: courtesy

Herbrand's Theorem (1930)

If a set of quantifier-free formulas is consistent, it has a simple 'Herbrand' model and, if it is not, its inconsistency will show up in some finite procedure.*

Hence: quantifier-free (sometimes artificial looking) axioms.

*equivalently (removing or putting $\forall$'s): universal sentences $(\forall x_1)\ldots(\forall x_n)Q(x_1,\ldots,x_n)$ with Q quantifier-free.

Notation

1. $\mathbb{N}$ consists of the (finite) positive integers.

2. In a term $\tau(\underbrace{x_1, \ldots, x_k}_{\vec{x}})$, $x_1, \ldots, x_k$ are the distinct free variables.

ERNA's language

(preview of meaning in [..])

- connectives: $\wedge, \neg, \vee, \rightarrow, \leftrightarrow$

- quantifiers: $\forall, \exists$

- an infinite set of variables

- 4 relation symbols:

  $=$ (binary)

  $\leq$ (binary)

  $\mathcal{I}$ (unary); notations for $\mathcal{I}(x)$: 'x is infinitesimal' or '$x \approx 0$'

  $\mathcal{N}$ (unary); notation for $\mathcal{N}(x)$: 'x is hypernatural'

- 5 individual constant symbols:

  0, 1,

  ω [infinite hypernatural], ε [$= 1/\omega$],

  $\uparrow$; notation 'x is undefined' for '$x =\uparrow$' [e.g. $1/0$ is undefined, $1/0 =\uparrow$]; notation 'x is defined' for '$x \neq\uparrow$'.

- function symbols:

  - (unary) abs.val. $|\ \ |$, ceiling $\lceil\ \ \rceil$, weight $\|\ \ \|$ [$\|\pm p/q\| = \max\{|p|, |q|\}$ for p and $q \neq 0$ relatively prime hypernaturals, else undefined]

  - binary $+, -, ., /, \hat{}$ [$x\hat{}n = x^n$ for hypernatural n, else undefined]

  - for each $k \in \mathbb{N}$, k k-ary function symbols $\pi_{k,i}$ ($i = 1, \ldots, k$) [i-th projection of a k-tuple $\vec{x}$]

- for each formula φ with $m+1$ free variables, without quantifiers or terms involving min, an m-ary function symbol $\min_\varphi$ [$\min_\varphi(\vec{x})$ = least hypernatural n with $\varphi(n, \vec{x})$; $= 0$ if there are none.]

- for each triple

  $(k, \sigma(x_1, \ldots, x_m), \tau(x_1, \ldots, x_{m+2}))$

  with $k \in \mathbb{N}$, σ and τ terms not involving min, an $(m+1)$-ary function symbol $\mathrm{rec}^k_{\sigma\tau}$ [function obtained from σ, τ by recursion, after the model

  $$f(0, \vec{x}) = \sigma(\vec{x}),$$
  $$f(n+1, \vec{x}) = \tau(f(n, \vec{x}), n, \vec{x})$$

  k restricts growth*]

*important for finitistic consistency proof

ERNA's Axioms

- axioms of first-order logic

- axioms for hypernaturals, including

  3. if x is hypernatural, then $x \geq 0$

  4. ω is hypernatural.

- definition: 'x is infinite' stands for '$x \neq 0 \wedge 1/x \approx 0$'; 'finite' stands for 'not infinite'; 'x is natural' stands for 'x is hypernatural and finite'.

- axioms for infinitesimals, including

  1. if $x \approx 0$ and $y \approx 0$, then $x + y \approx 0$

  2. if $x \approx 0$ and y is finite, then $x.y \approx 0$

6. $\varepsilon \approx 0$

7. $\varepsilon = 1/\omega$.

- field axioms [defined elements constitute an ordered field of characteristic zero with absolute value function] including

  $x + 0 = x$,

  $x + (0 - x) = 0$,

  if $x \neq 0$ then $x.(1/x) = 1$.

- Archimedean axiom: ...(easy)...

- theorem if x is defined, $\lceil x \rceil$ is the least hypernatural $\geq x$.

- power axioms: ...(easy)...

- projection axiom schema: ...(easy)...

- weight axioms: ...(artificial)...

- theorem If p and $q \neq 0$ are relatively prime hypernaturals*, then

$$\| \pm p/q\| = \max\{|p|, |q|\}.$$

  If x is not a hyperrational†, $\|x\|$ is undefined.

- theorem If $\|x\|$ and $\|y\|$ defined,

  $\|x + y\| \leq (1 + \|x\|)(1 + \|y\|)$,

  $\|x\hat{}y\| \leq (1 + \|x\|)\hat{}(1 + \|y\|)$, etc.

*involves quantifiers

†quantifier-free: $\mathcal{N}(p) \rightarrow \neg\mathcal{N}(p|x|)$

- theorem If $\tau(\vec{x})$ is a term not involving ω, ε, rec or min, then there exists a $k \in \mathbb{N}$ such that

$$\|\tau(\vec{x})\| \leq 2_k^{\|\vec{x}\|},$$

where

$$\|(x_1, \ldots, x_n)\| := \max\{\|x_1\|, \ldots, \|x_n\|\}$$

and

$$2_k^x := \underbrace{2\hat{}(\ldots 2\hat{}(2\hat{}(2\hat{}x)))}_{k\ 2\text{'s}}.$$

- recursion axioms For $k \in \mathbb{N}$, σ and τ not involving $\mathcal{I}$ or min:

  1.

  $$\mathrm{rec}^k_{\sigma\tau}(0, \vec{x}) = \sigma(\vec{x})$$
  $$\text{if } \sigma(\vec{x}) \text{ defined and } \|\sigma(\vec{x})\| \leq 2_k^{\|\vec{x}\|},$$

  undefined if $\sigma(\vec{x})$ undefined, 0 otherwise.

  2.

  $$\mathrm{rec}^k_{\sigma\tau}(n+1, \vec{x}) = \tau(\mathrm{rec}^k_{\sigma\tau}(n, \vec{x}), n, \vec{x})$$
  $$\text{if RHS defined and } \|\text{RHS}\| \leq 2_k^{\|\vec{x}, n+1\|},$$

  undefined if RHS undefined, 0 otherwise.

- axiom schema for internal minimum:
  . . . (artificial). . .

- theorem If φ does not involve $\mathcal{I}$ or min, and if there are hypernatural n's such that $\varphi(n, \vec{x})$, $\min_\varphi(\vec{x})$ is the least of these. If there are none, $\min_\varphi(\vec{x}) = 0$.

- corollary Proofs by hypernatural induction.

- axiom schema for external* minimum:
  . . . (artificial). . .

- theorem Let $\|\vec{x}\|$ be finite. If there are natural n's such that $\varphi(n, \vec{x})$, $\min_\varphi(\vec{x})$ is the least of these. If there are none, $\min_\varphi(\vec{x}) = 0$.

- corollary Proofs by natural induction.

*$\mathcal{I}$ allowed in φ

- axioms on (un)defined terms, including

  1. $0, 1, \omega, \varepsilon$ are defined

  2. x defined iff $\|x\|$ defined

  5. $x\frown y$ is defined iff x and y are defined and y is hypernatural.

  7. if x is not a hypernatural, $\mathrm{rec}^k_{\sigma\tau}(x, \vec{y})$ is undefined.

- theorem If x is defined, it is hyperrational.

Proof:

x defined iff $\|x\|$ defined *(part of axiom on (un)defined terms)*.

$\|x\|$ defined iff x hyperrational *(part of theorem)*.

$\square$

Remarks

- ‘Finitistic’ consistency proof within PRA (Primitive Recursive Arithmetic), a good approximation of Hilbert’s Program (scuttled by Gödel).

- ERNA has proof-theoretic strength of ERA (Elementary Recursive Arithmetic); hence the name ‘ERNA’.

- no standard part function, results up to infinitesimals. (‘An infinitesimal difference is as good as equality for physical purposes.’)

Applications (in NSM2004 Proceedings):

- sup-up-to-infinitesimals for sets $\{x \mid f(x) > 0\}$

- $\sqrt{x}$ (up to infinitesimals) for finite $x \geq 0$

ERNA + TRANSFER

Notations:

n, m, k = hypernatural variables.

'standard n' = finite hypernatural = in $\mathbb{N}$.

Abbreviation:

$$(\forall^{\text{st}} n)\varphi(n)$$

stands for ERNA's

$$\mathcal{N}(n) \wedge \neg\mathcal{I}(1/n) \to \varphi(n).$$

(Quantifier free, allowed in axiom below.)

- Transfer Axiom Schema

  For every quantifier-free formula $\varphi(n)$ not involving min, $\mathcal{I}, \omega$:*

$$\varphi(n) \vee (0 < \text{min}_{\neg\varphi} = \text{finite})$$

By Thm above, $\text{min}_{\neg\varphi}$ is either 0 or least counterexample to $\varphi(n)$. Hence TAS states

$$(\forall^{\text{st}} n \geq 1)\varphi(n) \rightarrow (\forall n \geq 1)\varphi(n)$$

without quantifiers (required for Herbrand's thm in consistency proof).

- Metatheorem ERNA+TAS has finitistic consistency proof. (Finite iteration of the one for ERNA.)

*$\text{min}_{\neg\varphi}$ excludes min, consistency proof excludes $\mathcal{I}, \omega$

- Corollary: ‘multivariable’ transfer

$$(\forall^{\text{st}} n \geq 1)(\forall^{\text{st}} m \geq 1)\varphi(n,m) \\ \rightarrow (\forall n \geq 1)(\forall m \geq 1)\varphi(n,m)$$

Proof: TAS + some kind of pairing functions

- Abbreviation: ‘x standard’ for ‘x rational’ ($\pm p/q$ with p and $q \neq 0$ naturals).

- Corollary: ‘general’ transfer

$$(\forall^{\text{st}} x)\varphi(x) \rightarrow (\forall x)\varphi(x)$$

i.e. ERNA’s

$$(x \text{ rational} \rightarrow \varphi(x)) \rightarrow (x \text{ defined} \rightarrow \varphi(x))$$

Proof: *multivariable transfer + any defined x is hyperrational.*

Applications

- characterization of Cauchy hypersequence (not involving min, $\omega, \mathcal{I}$):

$$(\forall^{\text{st}} k)(\exists^{\text{st}} N)(\forall^{\text{st}} n, m \geq N)(|s_n - s_m| < \frac{1}{k})$$

$$\Longleftrightarrow$$

$$s_n \approx s_m \text{ for all infinite } m, n$$

- convergence-up-to-infinitesimals of Cauchy hypersequences (not involving...) to any infinitely indexed term.

- characterization of continuity (f not involving...):

$$(\forall^{\text{st}} x)(\forall^{\text{st}} k)(\exists^{\text{st}} N)(\forall^{\text{st}} y)$$

$$(|x - y| < \frac{1}{N} \rightarrow |f(x) - f(y)| < \frac{1}{k})$$

$$\Longleftrightarrow$$

$$f(x) \approx f(y) \text{ for all } x \approx y$$

- sup-up-to. . . of increasing bounded hyper-sequences (not involving. . .):

$$s_1 \leq s_2 \leq \cdots \leq M = \text{finite}$$

has s_ω as sup-up-to-infinitesimals*

- sup-up-to... principle: $\{x \mid \varphi(x)\}$ (φ quantifier-free, not involving. . .) nonempty and finitely bounded above has sup-up-to-infinitesimals (highly nontrivial)

*sup is beyond PRA's strength

GENERALIZING TRANSFER

compare

$$(\forall^{\mathrm{st}} x)(0 < x < 1 \to \frac{1}{x} > 1) \qquad \text{true}$$

$$(\forall x)(0 < x < 1 \to \frac{1}{x} > 1) \qquad \text{true}$$

to*

$$(\forall^{\mathrm{st}} x)(0 < x < 1 \to \frac{1}{x} > 1 + \frac{1}{\omega}) \qquad \text{true}$$

$$(\forall x)(0 < x < 1 \to \frac{1}{x} > 1 + \frac{1}{\omega}) \quad \text{false for } \frac{\omega}{1+\omega}$$

$$(\forall x)(0 < x < 1 \to \frac{1}{x} \gtrapprox 1 + \frac{1}{\omega}) \qquad \text{true}$$

*ω disallowed in TAS

Notation: to get $\widetilde{\varphi}$ from φ: replace $<$ with $\lessapprox$ unless between rationals.

- theorem For φ quantifier free, not involving...+ conditions:

  $$(\forall^{\text{st}} x)\varphi(x) \to (\forall x)\widetilde{\varphi}(x)$$

  Remark: φ may contain ω in terms like

  $$\pi_\omega := 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots - \frac{1}{2\omega + 1}\right)$$

  $$\sin_\omega(x) := \sum_{n=0}^{\omega} (-1)^n \, \frac{x^{2n+1}}{(2n+1)\,!}$$

- example:

  $$(\forall^{\text{st}} n)(s_n \leq s_\omega = \text{finite}) \to (\forall n)(s_n \lessapprox s_\omega)$$

Future Research/Work in Progress

- Π_2-transfer

$$(\forall^{\mathrm{st}} x)(\exists^{\mathrm{st}} y)\varphi(x,y) \rightarrow (\forall x)(\exists y)\varphi(x,y)$$

- Saturation (provable in ERNA? consistent with ERNA?)

- Bolzano-Weierstrass theorem,...