Applications of S-measurability to regularity and limit theorems
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1966 Robinson: *defined S-measure 0*, used Egoroff's Theorem to prove that for a sequence f_n of measurable functions, the complement of a set characterizing uniform convergence of f_n has S-measure 0.

1981 Henson,Wattenberg: Characterized S-measure in general, showed independently that the set above has S-measure 0; Egoroff Theorem was easy corollary.

2002,4 R: Extended S-measurability, applied to sets and functions on non-topological measure space. (Theorems of Riesz, Radon-Nikodym)

Today Application to regularity and limits, including Birkhoff Ergodic Theorem

1 Loeb measures, S-measures

Suppose X a set, $\mathcal{A}$ is an algebra on X

Two natural algebras on ${}^*X$:

${}^*\mathcal{A}$ (=internal subsets of ${}^*X$)

$\mathcal{A}_0 = \{{}^*A : A \in \mathcal{A}\}$ (="standard sets")

These lead to two distinct σ-algebras:

$$\mathcal{A}_S = \text{the smallest } \sigma-\text{algebra containing } \mathcal{A}_0$$

$$\mathcal{A}_L = \text{the smallest } \sigma-\text{algebra containing } {}^*\mathcal{A}$$

(both normally external)

Recall:

If μ is a (finitely-additive) finite measure on $(X, \mathcal{A})$ then

$$^*\mu : {}^*\mathcal{A} \longrightarrow^* [0, \infty)$$
$$^{\circ *}\mu : {}^*\mathcal{A} \longrightarrow [0, \infty)$$

$({}^*X, {}^*\mathcal{A}, {}^{\circ *}\mu)$ is an external, standard, f.a. finite measure space.

${}^{\circ *}\mu$ extends to a σ-additive measure μ_L on $({}^*X, \mathcal{A}_L)$ (the *Loeb space*).

Of course,

1. We can also do this with any internal finitely-additive *measure, not just those arising from standard measures.
2. μ_L is also a standard measure on $\mathcal{A}_S$

2 Properties of S-measures

1. $\forall E \in \mathcal{A}_S$,

$$\begin{aligned}\mu_L(E) &= \inf\{\mu(A) \;:\; E \subseteq {}^*A, A \in \mathcal{A}\} \\ &= \sup\{\mu(A) \;:\; {}^*A \subseteq E,\; A \in \mathcal{A}\} \\ &= \mu(\underbrace{X \cap E}_{:=S(E)})\end{aligned}$$

2. If $f : X \longrightarrow \mathbb{R}$ is $\mathcal{A}$-measurable, then ${}^{\circ *}f : {}^*X \longrightarrow \mathbb{R}$ is $\mathcal{A}_S$ -measurable
3. If $G : {}^*X \longrightarrow \mathbb{R}$ is $\mathcal{A}_S$ -measurable, and $g = G|_X$, then
   (a) $g : X \longrightarrow \mathbb{R}$ is $\mathcal{A}$-measurable,
   (b) $\mu_L(\{x \in {}^*X : {}^*g(x) \not\approx G(x)\}) = 0$
   (c) For any $p > 0$, $G \in \mathcal{L}^p(\mu_L) \Leftrightarrow g \in \mathcal{L}^p(\mu)$ (with same integral)

Remarks:

(a) S-measurability should be useless.

(b) It seems to be a genuinely useful tool for applying Loeb measure methods to nontopological measure spaces.

3 Regularity

Theorem 1. *Let $(X, \mathcal{B}, \mu)$ be a finite Borel measure on a Polish space (that is, X is a complete separable metric space, and $\mathcal{B}$ is the Borel σ − algebra on X). Then μ is Radon (= compact inner-regular).*

PROOF:

Fix a countable dense subset Γ of X.

If E is any closed subset of X, put

$$E' = \bigcap_{\epsilon \in \mathbb{Q}^+} \bigcup \{ {}^*B(\gamma, \epsilon) : \gamma \in \Gamma, B(\gamma, \epsilon) \cap E \neq \emptyset \}$$

Note: $E' \in \mathcal{B}_S$

Exercise: $E' = st^{-1}(E)$ (Hint: for $\subseteq$, use completeness.)

Cor: For *every* $E \in \mathcal{B}$, $st^{-1}(E) \in \mathcal{B}_S$.

Let $E \in \mathcal{B}$ and $\epsilon > 0$.

$\mu_L(st^{-1}(E)) = \mu(X \cap st^{-1}(E)) = \mu(E)$

$\exists A \in \mathcal{B}$ with ${}^*A \subseteq st^{-1}(E)$ and $\mu(A) \geq \mu(E) - \epsilon$

Put $K = st({}^*A)$, note K is compact, $A \subseteq K \subseteq E$.

Therefore $\mu(K) \geq \mu(A) > \mu(E) - \epsilon$

4 Limits

EG

Lemma 1. *(Fatou) Let* $f_n \geq 0$ *be a sequence of measurable functions on a finite measure space* $(X, \mathcal{A}, \mu)$*. Put* $\underline{f} = \lim_{N\to\infty} \underbrace{\inf_{n\geq N} f_n}_{f^N}$ *Then* $\int \underline{f} d\mu \leq {}^{\circ *}\int f_H d^*\mu$ *for any infinite* H

PROOF: Put

$$E = \{x \in {}^*X | {}^{\circ *}\underline{f}(x) = \underbrace{\lim_{N\to\infty} \inf_{n\geq N}}_{\text{(standard indices)}} {}^{\circ *}f_n\}$$

Note $E \in \mathcal{A}_S$ and $S(E^{\complement}) = \emptyset$, so $\mu_L(E^{\complement}) = 0$

Let $M, \epsilon > 0$ standard; then for any $x \in E$ there is a standard N with

$$\max\{{}^*\underline{f}(x), M\} \leq {}^*f^N(x) + \epsilon \leq f_H(x) + \epsilon$$

Then:

$$\begin{aligned}\int \max\{{}^{\circ *}\underline{f}(x), M\}d\mu_L &= \int_E {}^{\circ}\max\{{}^{*}\underline{f}(x), M\}d\mu_L \\ &\leq \int_E {}^{\circ}f_H(x)d\mu_L + \epsilon\mu(X) \\ &\leq \int {}^{\circ}f_H(x)d\mu_L + \epsilon\mu(X)\end{aligned}$$

Since $\max\{{}^{\circ *}\underline{f}(x), M\}$ is S-measurable, we can restrict to X, let $M \to \infty$, then let $\epsilon \to 0$, and obtain

$$\int \underline{f}d\mu \leq \int {}^{\circ}f_H(x)d\mu_L$$

and this last term is either ${}^{\circ *}\int f_H d^*\mu$ (if f_H is S-integrable) or ∞ (if not).

Either way, this proves the inequality in the Theorem.

5 Ergodic Theorem

Kamae(1982): Essentially new nonstandard proof of Ergodic Theorem:

Theorem 2. *Let $(X, \mathcal{A}, \mu)$ be a probability space, $T : X \longrightarrow X$ measure preserving, and $f \in \mathcal{L}^1(\mu)$. Then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$ exists almost surely, and the integral of this limit is $\int f d\mu$*

Used deep von Neumann-Maharam structure theory to represent general dynamical system as a factor of a hyperfinite Loeb space with the canonical internal transformation.

Later 'standardized' (Katznelson, Weiss; McKean)

Problem: find other applications of the representation.

Remainder of lecture: 'wrong' solution - use S-measurability to eliminate the Kamae representation, retain the essentially nonstandard nature of his proof.