On orthogonal Lévy martingales and Malliavin calculus based on chaos

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Horst Osswald

1. Introduction

(i) (Tom Lindstrøm), Hyperfinite Lévy processes, *Stochastics* 2004.

(ii) (Nigel Cutland, Siu-Ah Ng), A nonstandard approach to the Malliavin calculus, Conference Volume 1995

(iii) (HO) Malliavin calculus for product measures on $\mathbb{R}^{\mathbb{N}}$ based on chaos, *Stochastics* 2005.

In article (iii) Malliavin calculus is developed for the product measure $\mu^\infty = \mu$ on $\mathbb{R}^{\mathbb{N}}$ derived from an arbitrary Borel probability measure μ^1 on $\mathbb{R}$. We obtain Malliavin calculus for arbitrary abstract Wiener spaces over "little" l_2.

Our aim is to extend the techniques and results of paper (iii) to the the space $\mathbb{R}^{[0,\infty[}$, which is the space of càdlàg functions, endowed with the Skorohod topology.

Therefore, we replace $\mathbb{R}^{\mathbb{N}}$ by ${}^*(\mathbb{R}^{\mathbb{N}})$ in an $\aleph_1$-saturated model W of mathematics and fix an $H \in {}^*\mathbb{N}$, $H \approx \infty$. Then we may identify

$$\left\{ \frac{1}{H}, \frac{2}{H}, ... \right\} \equiv [0, \infty[,$$

because there is a close relationship between the Loeb measure on $\frac{{}^*\mathbb{N}}{H}$ and Lebesgue measure on $[0, \infty[$.

Examples:

(I) (Lindstrøm) Let $L : \Omega \times {}^*\mathbb{N} \rightarrow {}^*\mathbb{R}$ be an internal Lévy process on an internal probability space Ω such that we have $\widehat{P}$-almost surely:

$$\frac{n}{H} \not\approx \infty \Rightarrow |L(\cdot, n)| \not\approx \infty.$$

Then ${}^\circ L : \Omega \times [0, \infty[\rightarrow \mathbb{R}$, defined by

$$({}^\circ L)(\cdot, r) := \lim_{{}^\circ \frac{n}{H} \downarrow r} {}^\circ (L(\cdot, n)),$$

is a càdlàg Lévy process. Vice versa, each Lévy process can be essentially obtained in this way.

We may assume that $\Omega = {}^*\left(\mathbb{R}^{\mathbb{N}}\right)$ and the measure on Ω is the product $\mu = \mu^\infty$ of a certain internal probability measure μ^1 on ${}^*\mathbb{R}$. Moreover, we may assume that $L(X, n) = \sum_{i=1}^{n} X_i$.

(II) (Cutland) Brownian motion: let

$$d\mu^1 := e^{-\frac{H}{2}x^2} dx \sqrt[2]{\frac{H}{2\pi}}.$$

Then $^\circ\frac{n}{H} \mapsto {}^\circ(X_1 + ... + X_n)$ is a continuous Brownian motion.

Cutland and Ng studied Malliavin calculus using this construction.

(III) Poisson processes: let

$$\mu^1(B) := \sum_{i \in {}^*\mathbb{N}_0 \cap B} e^{-\frac{1}{H}} \frac{(\frac{1}{H})^i}{i!}.$$

Then $^\circ(X_1 + ... + X_n)$ is a Poisson process.

Pioneers: R. Anderson, J. Keisler, D. Hoover and E. Perkins, T. Lindstrøm.

2. General assumptions

(H$_0$) There exists a Borel measurable internal mapping $\mathfrak{E} : {}^*\mathbb{R} \to {}^*\mathbb{R}$ with the following properties

(a) $\mathfrak{E}$ is bijective outside of internal sets D, R in the domain and range of $\mathfrak{E}$ and S-bounded. We assume that $\mu^1(D) \approx 0 \approx \mu^1(R)$.

(b) $H \cdot \mathbb{E}_{\mu^1}\mathfrak{E}^2$, $H \cdot \mathbb{E}_{\mu^1}\mathfrak{E}$ are limited.

(c) $H \cdot \mathbb{E}_{\mu^1}\mathfrak{E}^2 \not\approx 0$.

Then $\mathbb{E}_\mu \left|\sum_{i=1}^N \mathfrak{E}\left(X_i\right)\right|^2 \not\approx \infty$ for all N such that $\frac{N}{H} \not\approx \infty$. By Condition (c), the Lévy process ${}^\circ L^{\mathfrak{E}}$ is not identical to 0. Here

$$L^{\mathfrak{E}}(X, n) := \sum_{i=1}^{n} \mathfrak{E}\left(X_i\right).$$

Siu-Ah Ng shows that Condition H_0 is not an essential restriction.

In many examples, including Brownian motion and Poisson processes, the following Condition (H) is fulfilled for μ^1.

We will study stochastic integration and Malliavin calculus for standard Lévy processes, resulting from measures μ^1, satisfying Condition (H).

Condition (H): There exists a Borel measurable mapping $\mathfrak{E} : {}^*\mathbb{R} \to {}^*\mathbb{R}$ such that (H_0) is true and

$$H\mathbb{E}_{\mu^1}\left(x - \mathfrak{E}\left(x\right)\right)^2 \approx 0 \approx H\mathbb{E}_{\mu^1}\left(x - \mathfrak{E}\left(x\right)\right)$$

Examples: In the cases of BM and PP, we have

$$\mathfrak{E}\left(x\right) := 1_{\{|x|\leq 1\}}(x) \cdot x.$$

It follows that

$$\mathbb{E}_\mu e^{\lambda|\mathfrak{E}(X_1)+...+\mathfrak{E}(X_H)|} \not\approx \infty$$

for some $\lambda \in \mathbb{R}^+$ (Protter, Lindstrøm). By results of

S. Boucheron, G. Lugosi, M. Massart, Concentration inequalities using the entropy method, *The Annals of Probability* 2003,

functions of the form

$$F(1) \cdot \mathfrak{E}(X_1) + ... + F(H) \cdot \mathfrak{E}(X_H)$$

have exponential moments if F is S-bounded.

Siu-Ah Ng found a different proof of this fact for the measures of his smaller equivalence class of measures leading to the same Lévy process.

3. Orthogonal polynomials

Using a slight modification of the Gram Schmidt orthogonalization procedure applied to $1, \mathfrak{E}, \mathfrak{E}^2, ...$, we construct a sequence $(p_i)_{i\in\mathbb{N}_0}$ as follows: set $p_0(x) := 1$. The number 0 is called an **uncritical** exponent. Define $p_1 := \mathfrak{E} - \mathbb{E}_{\mu^1}\mathfrak{E}$. We have

$$H\left\|p_1\right\|_2^2 = H\mathbb{E}_{\mu^1}p_1^2 \not\approx 0.$$

The number 1 is called an **uncritical** exponent.

Assume that $p_0, ..., p_{n-1}$ are already defined and that $0 = u_0 < ... < u_l \leq n-1$ are the uncritical exponents below n. Define

$$p_n := \mathfrak{E}^n - \sum_{i=0}^{l} \frac{\mathbb{E}_{\mu^1}\left(\mathfrak{E}^n \cdot p_{u_i}\right)}{\left\|p_{u_i}\right\|_2^2} p_{u_i}.$$

If $H \cdot \left\|p_n\right\|_2^2 \approx 0$, then n is called **critical**, otherwise n is called **uncritical**.

Why the notion "uncritical"? Suppose that u is uncritical. Then

$$\left|\frac{\mathbb{E}_{\mu^1}\left(\mathfrak{E}^n \cdot p_u\right)}{\|p_u\|_2^2}\right| = \left|\frac{H \cdot \mathbb{E}_{\mu^1}\left(\mathfrak{E}^n \cdot p_u\right) \not\approx \infty}{H \cdot \|p_u\|_2^2 \not\approx 0}\right| \not\approx \infty$$

Examples

We denote the set of uncritical exponents $n \geq 1$ by $\mathbb{N}_L$.

(1) In the cases of Brownian motion and Poisson processes $\mathbb{N}_L = \{1\}$.

(2) For each Borel set $B \subseteq {}^*\mathbb{R}$ set

$$\mu^1(B) := \int_B \frac{\sqrt{2H}}{1 + H^2 x^4} \frac{1}{\pi} dx.$$

For $\mathfrak{E}\left(x\right) := \mathbf{1}_{\{|x| \leq 1\}}(x) \cdot x$ our condition (H) is fulfilled and $\mathbb{N}_L = \{1\}$.

4. An example, where polynomials of arbitrary degree appear

Let

$$\mu^1(B) := \int_{{}^*[-1,1]\cap B} \frac{H}{1+(Hx)^2} dx \frac{1}{\pi}.$$

Then $H\mathbb{E}_{\mu^1} x^{2n} \approx \frac{2}{2n-1}$ and all positive integers are uncritical.

Siu-Ah Ng characterized the measures having uncritical exponents of arbitrary degree by means of the associated Lévy measure.

5. Stochastic Integration

Let

$$f : \Omega \times [0, \infty[\to \mathbb{R}$$

be a non-time-anticipating process in $L^2(\widehat{\mu} \otimes \lambda)$ and let

$$F : \Omega \times {}^*\mathbb{N} \to {}^*\mathbb{R}$$

be a non-time-anticipating S-square integrable lifting of f.

Non-time-anticipating means that F is $(\mathcal{B}_{t-1})_{t \in {}^*\mathbb{N}}$-adapted, where $(\mathcal{B}_t)_{t \in {}^*\mathbb{N}}$ is the **natural filtration** on $\Omega = {}^*\left(\mathbb{R}^{\mathbb{N}}\right)$.

Non-time-anticipating of f means that f is non-time-anticipating with respect to the standard part $(\mathfrak{b}_t)_{t \in [0,\infty[}$ of $(\mathcal{B}_{t-1})_{t \in {}^*\mathbb{N}}$, constructed by Jerry Keisler.

Now fix an uncritical $k \in \mathbb{N}$. Then the **k-th integral** $\int f dp_k$ **of** f, is defined by setting for each $r \in [0, \infty[$

$$\int f dp_k(X,r) := \lim_{^\circ \frac{s}{H} \downarrow r} {}^\circ \sum_{t \leq s} F(X,t) p_k(X_t).$$

Thus, we first integrate internally with respect to the discrete martingale $X \mapsto \left(\sum_{t \leq n} p_k(X_t)\right)_{n \in {}^*\mathbb{N}}$ and then take the standard part.

In order to obtain the integral independent of k, we integrate sequences $(f_k)_{k \geq 1, k \text{ is uncritical}}$ of non-time-anticipating processes f_k such that

$$\sum_{k \in \mathbb{N}_L} \int_{\Omega \times [0,\infty[} f_k^2 d\widehat{\mu} \otimes \lambda < \infty,$$

setting for $r \in [0, \infty[$

$$\int (f_k)\, dp\, (\cdot, r) := \sum_{k \in \mathbb{N}_L} \int f_k dp_k(\cdot, r).$$

6. Multiple Integrals

We integrate deterministic functions

$$f : \mathbb{N}_L^n \times [0,\infty[^n \rightarrow \mathbb{R}$$

such that $\sum_{k \in \mathbb{N}_L^n} \int_{[0,\infty[^n} f^2 (k, \cdot)\, d\lambda^n < \infty$. f is called **symmetric** [**λ^n-a.s.**] if

$$f(k_1, ..., k_n, t_1, ..., t_n) = f(k_{\sigma_1}, ..., k_{\sigma_n}, t_{\sigma_1}, ..., t_{\sigma_n})$$

for all permutation σ on $\{1, ..., n\}$ and [λ^n-almost] all $t_1, ..., t_n$.

Let us first define for suitable liftings F of f and fixed $(k_1, ..., k_n) \in \mathbb{N}_L^n$

$$I_{(k_1,...,k_n)}(f(k_1, ..., k_n, \cdot))(X) :=$$
$$^\circ \sum_{t_1<...<t_n \in {}^*\mathbb{N}} F(k_1, ..., k_n, t_1, ..., t_n) p_{k_1}(X_{t_1}) \cdot ... \cdot p_{k_n}(X_{t_n}).$$

similar to the work of Cutland and Ng for the classical Wiener space.

In order to define $I_n(f)$ independent of $k \in \mathbb{N}_L^n$, we set

$$I_n(f) := \sum_{(k_1,...,k_n) \in \mathbb{N}_L^n} I_{(k_1,...,k_n)}(f_{(k_1,...,k_n)}).$$

7. Chaos decomposition

Let $\mathcal{W}$ be the sub-σ-algebra of the Loeb σ-algebra $L_{\mu}(\mathcal{B})$, generated by the "multiple" integrals $I_{(k)}(f)$ with $k \in \mathbb{N}_L$, $f \in L^2(\lambda)$, augmented by the $\widehat{\mu}$-nullsets.

Each polynomial $Q(I_{(k)}(f))$ in $I_{(k)}(f)$ is a linear combination of multiple integrals with kernels of the form $f_1 \odot \ldots \odot f_n$.

THEOREM 7.1. *Each $\varphi \in L^2_{\mathcal{W}}(\widehat{\mu})$ has the decomposition*

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n) = \sum_{n=0}^{\infty} I_n({}^{\circ}F_n) =$$

$$\sum_{n=0}^{\infty} \sum_{\overrightarrow{k} \in \mathbb{N}_L^n} {}^{\circ} \sum_{t_1<\ldots<t_n} F_n\left(\overrightarrow{k}, \overrightarrow{t}\right) \cdot p_{k_1}(X_{t_1}) \cdot \ldots \cdot p_{k_n}(X_{t_n})$$

In order to obtain the Clark Ocone formula, note that the preceding term equals

$$\mathbb{E}_{\widehat{\mu}}\varphi + \sum_{n=1}^{\infty} \sum_{k \in \mathbb{N}_L} \sum_{\overrightarrow{k} \in \mathbb{N}_L^{n-1}} {}^{\circ}\sum_{t \in {}^*\mathbb{N}}$$

$$\overbrace{\underbrace{\left(\sum_{t_1 < \ldots < t_{n-1} < t} F_n \left(\overrightarrow{k}, k, \overrightarrow{t}, t \right) p_{k_1}(X_{t_1}) \cdot \ldots \cdot p_{k_{n-1}}(X_{t_{n-1}}) \right)}_{}}^{\mathbb{E}^{\mathcal{B}_{t^-}} \sum_{t_1 < \ldots < t_{n-1}} F_n \left(\overrightarrow{k}, k, \overrightarrow{t}, t \right) p_{k_1}(X_{t_1}) \cdot \ldots \cdot p_{k_{n-1}}(X_{t_{n-1}})} p_k(X_t) =$$

$$\mathbb{E}\varphi + \int \left(t \mapsto \mathbb{E}^{\mathcal{B}_{t^-} \vee N_{\widehat{\mu}}} \sum_{n=1}^{\infty} I_{n-1} {}^{\circ}F_n\left(\cdot, k, \cdot, t\right) \right)_{k \in \mathbb{N}_L} dp.$$

8. Comparison with the standard literature

Schoutens (2000) starts with the power jump process

$$L_t^{(i)} := \sum_{0<s\leq t} (\Delta L_s)^i$$

of a Lévy process L such that the associated Lévy measure has exponential moments. Then he uses the Lévy martingales

$$Y_t^{(i)} := L_t^{(i)} - \mathbb{E}L_t^{(i)}$$

to define multiple integrals. The integrators of these are orthogonal martingales $Z_t^{(i)}$, where $Z_t^{(i)}$ is a linear combination of the $Y_t^{(j)}, j \leq i$. Schoutens uses these multiple integrals to prove chaos decomposition for the L^2-functions on the underlying probability space, which are measurable with respect to the σ-algebra, generated by the Lévy process L.

Øksendal, Di Nunno, Proske et al. also used two parameter processes, now depending on $[0, \infty[$ and on the whole real numbers.

The difference between his work and our approach is, roughly speaking, the following: we orthonormalize the powers

$$1 = (\Delta L_t)^0, (\Delta L_t)^1, (\Delta L_t)^2 ...$$

of the increments ΔL_t to

$$p_0, p_1, p_2, ...$$

independent of t and integrate first for $k \geq 1$ with respect to the martingales

$$\sum_{s \leq t} p_k (\Delta L_s).$$

Similar to Cutland's and Ng's work for the Brownian motion case, we have here also a nice recipe for the computations of the kernels of the chaos decomposition

$$\varphi = \sum_{i=0}^{\infty} I_n({}^{\circ}F_n).$$

If Φ is an S-square integrable lifting of φ, then

$$F_n(k_1, ..., k_n, t_1, ..., t_n) = \mathbb{E}_{\mu}\Phi \cdot p_{k_1}(X_{t_1}) \cdot ... \cdot p_{k_n}(X_{t_n}) \cdot H^n =$$

$$\mathbb{E}_{\mu}\Phi \cdot \frac{\Delta M_{t_1}^{k_1}}{\Delta t_1} \cdot ... \cdot \frac{\Delta M_{t_n}^{k_n}}{\Delta t_n}.$$

9. Malliavin derivative

Let $\varphi \in L^2_{\mathcal{W}}(\widehat{\mu})$ with decomposition $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$.

The **Malliavin derivative** D is a densely defined operator on $L^2_{\mathcal{W}}(\widehat{\mu})$ in $\left(L^2_{\mathcal{W}}(\widehat{\mu} \otimes \lambda)\right)^{\mathbb{N}_L}$. This is the space of sequences $(\psi_k)_{k\in\mathbb{N}_L}$of $\psi_k \in L^2_{\mathcal{W}}(\widehat{\mu} \otimes \lambda)$ such that

$$\sum_{k\in\mathbb{N}_L} \int_{\Omega\times[0,\infty[} \varphi_k^2 d\widehat{\mu} \otimes \lambda < \infty.$$

For $k \in \mathbb{N}_L$ and $r \in [0, \infty[$ we define

$$(D\varphi)_k(\cdot, r) := \sum_{n=1}^{\infty} I_{n-1}(f_n(\cdot, k, \cdot, r)$$

for those φ such that $D\varphi$ converges in $\left(L^2_{\mathcal{W}}(\widehat{\mu} \otimes \lambda)\right)^{\mathbb{N}_L}$, in which case φ is called **Malliavin differentiable** .

10. The Skorohod integral

The **Skorohod integral** is a densely defined operator

$$\delta : \left(L^2_{\mathcal{W}}\left(\widehat{\mu} \otimes \lambda\right)\right)^{\mathbb{N}_L} \to L^2_{\mathcal{W}}\left(\widehat{\mu}\right).$$

In order to define δ we need a suitable decomposition of $(\varphi_k)_{k\in\mathbb{N}_L} \in \left(L^2_{\mathcal{W}}\left(\widehat{\mu} \otimes \lambda\right)\right)^{\mathbb{N}_L}$:

$$\varphi_k\left(\cdot, r\right) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, k, \cdot, r)).$$

Then

$$\delta\left(\varphi_k\right)_{k\in\mathbb{N}_L} := \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f_n})$$

for those $(\varphi_k)_{k\in\mathbb{N}_L}$ such that $\sum_{n=0}^{\infty} I_{n+1}(\widetilde{f_n})$ converges in $L^2_{\mathcal{W}}\left(\widehat{\mu}\right)$, in which case $(\varphi_k)_{k\in\mathbb{N}_L}$ is called **Skorohod integrable**.

The Skorohod integral is an extension of the integral, defined above, and it is the adjoint operator of the Malliavin derivative:

$$\left\langle (\psi_k)_{k\in\mathbb{N}_L}, D\varphi \right\rangle_{(L^2_{\mathcal{W}}(\widehat{\mu}\otimes\lambda))^{\mathbb{N}_L}} = \left\langle \delta\,(\psi_k)_{k\in\mathbb{N}_L}, \varphi \right\rangle_{L^2_{\mathcal{W}}(\widehat{\mu})}$$

if $\delta\,(\psi_k)_{k\in\mathbb{N}_L}$ and $D\varphi$ exist.