Infinitesimal Fourier transformation for the space of functionals

in

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ed. Matsushita, Hashimoto

background

M : 3 dim. or
4 dim.
manifold.

a construction of an invariant

$$\begin{tikzcd} V \arrow[d, "SU(2)"] \\ M \end{tikzcd}$$

$$\boxed{\left\{ \nabla : \text{connections on } V \text{ satisfying some P.D.E} \right\} \Big/ \text{Gauge}}$$

finite dimensional

$$\Downarrow$$

invariants

Originally, in Physics.

$$\boxed{\left\{ \nabla : \text{connections on } V \text{ satisfying some P.D.E.} \right\}}$$

infinite dimensional

NO PARALLELIZABLE

Borel measure

$$\int \mathcal{D}\nabla \, e^{i S_{\text{Lagrangian}}}.$$

Feynman path integral.

Problem.

Construct

a Fourier transformation

theory

for

functionals.

a functional

$$f : \{ a : \mathbb{R} \to \mathbb{R} \} \to \mathbb{C}.$$

$$(-\infty, \infty) \to (-\infty, \infty) \qquad \infty$$

3 types of "∞"

$$\{\infty, \infty, \infty\}$$

$$\Downarrow$$

different infinities.

nonstandard argument

instead of

standard argument

Relative set theory.

G. Wallet, Y. Peraire,
E. Gordon
K. Hrbacek

Λ : an infinite set

(for example $\mathbb{N}$)

$$\mathcal{F}_0(\Lambda) := \{\Lambda \setminus S \mid S : \text{a finite set}\}$$

Frechét filter

$\overline{\mathcal{F}_0(\Lambda)}$: Ultrafilter $\supset \mathcal{F}_0(\Lambda)$.

$$\begin{cases} ① \ \Lambda \in \overline{\mathcal{F}_0(\Lambda)}, \quad \phi \notin \overline{\mathcal{F}_0(\Lambda)}, \\ ② \ A, B \in \overline{\mathcal{F}_0(\Lambda)} \Rightarrow A \cap B \in \overline{\mathcal{F}_0(N)}, \\ ③ \ A \in \overline{\mathcal{F}_0(\Lambda)}, A \subset B \Rightarrow B \in \overline{\mathcal{F}_0(N)}, \end{cases}$$

$$④ \ A \subset \Lambda \Rightarrow A \text{ or } \Lambda \setminus A \in \overline{\mathcal{F}_0(\Lambda)}.$$

$$\boxed{\bigcap_{A \in \overline{\mathcal{F}_0(\Lambda)}} A = \phi}.$$

S : a set

$${}^*S = S^\wedge/\sim ,$$

$(s_\lambda), (t_\lambda) \in S^\wedge,$

$$(s_\lambda) \sim (t_\lambda) \overset{\text{def}}{\iff} \underline{\{\lambda \mid s_\lambda = t_\lambda\}} \in \mathcal{F}_0(\Lambda).$$

$\Uparrow$ true (or false)

Example

$${}^*\mathbb{N},\ {}^*\mathbb{Z},\ {}^*\mathbb{R},\ {}^*\mathbb{C},\ \cdots .$$

- Fourier series

1972. Wilhelmus A.J. Luxemburg

$$\sum_{k=0}^{\infty} a_k e^{2\pi i k x} \to \sum_{k=0}^{N} a_k e^{2\pi i k x}$$

an infinite number

- Fourier transformation

1988, '90 Moto-o Kinoshita

1989 E. I. Gordon independently

H : an infinite,
even,
positive number
$\in {}^*\mathbb{N}$.

$\varepsilon := \frac{1}{H}$, infinitesimal.

$$L := \{ \varepsilon z \in \varepsilon\, {}^*\mathbb{Z} \mid -\frac{H}{2} \leq \varepsilon z < \frac{H}{2} \}.$$

$$\{ a : L \to {}^*\mathbb{C},\ \text{internal} \}$$

$$a = [a_\lambda].$$

$$\delta(x) = \begin{cases} H, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

1962
Gaishi Takeuti

$$(Fa)(y) \overset{\text{def}}{=} \sum_{x \in L} \varepsilon \exp(-2\pi i x y)\, a(x),$$

$$(\bar{F}a)(y) \overset{\text{def}}{=} \sum_{x \in L} \varepsilon \exp(2\pi i x y)\, a(x).$$

Theorem (Kinoshita, Gordon)

(i) $F1 = \delta$, $F\delta = 1$,

$\overline{F}1 = \delta$, $\overline{F}\delta = 1$,

(ii) $F\overline{F} = \overline{F}F = id$,

$F^4 = id$,

(iii) $\langle a, b \rangle = \sum_{x \in L} \varepsilon \overline{a(x)} b(x)$,

F is unitary w.r.t. $\langle \cdot \rangle$.

$$(a*b)(x) := \sum_{y \in L} \varepsilon\, a(x-y)\, b(y),$$

(iv) $F(a*b) = F(a)F(b)$,

$F(ab) = F(a) * F(b)$.

by Gordon,

$\alpha \in (L^2(\mathbb{R}) \cap)\, C_0^\infty(\mathbb{R})$,

$a := {}^*\alpha|_L$

$\Rightarrow$

$$Fa \sim \mathcal{F}\alpha.$$

nonstandard standard

$$\mathbb{R} \to {}^*\mathbb{R} \to {}^*({}^*\mathbb{R})$$

many kinds of infinities,

many stages of infinities.

Λ_2 : an infinite set,

$\mathcal{F}_2 := \overline{\mathcal{F}_0(\Lambda_2)}$,

ultrafilter on Λ_2,

$\mathcal{F}_0(\Lambda_2) \subset \overline{\mathcal{F}_0(\Lambda_2)}$,

$$\left(\bigcap_{A \in \mathcal{F}_2} A = \phi \right).$$

$$\mathbb{R} \qquad {}^*\mathbb{R} \qquad {}^{\star}({}^*\mathbb{R})$$

$\downarrow$ $\qquad$ $\downarrow$

each element < a positive infinite ,

$\downarrow$

each element < a positive infinite

$${}^{*}\Lambda_2 \,, \qquad {}^{*}\mathcal{F}_2 \,.$$

$${}^{*}({}^{*}S) := ({}^{*}S)^{{}^{*}\Lambda_2} / {}^{*}\mathcal{F}_2 \,.$$

$$\boxed{H' \in {}^{*}({}^{*}\mathbb{N}) \,,}$$

an infinite,
positive,
even number

$$\boxed{\varepsilon' = \frac{1}{H'}} \,.$$

$$\Lambda = \mathbb{N}$$

$$[n] = [1, 2, 3, \cdots]$$

$$[n^2] = [1, 2^2, 3^2, \cdots]$$

$$\mathbb{R} \ni r \longmapsto \underset{\underset{{}^*r}{\parallel}}{[r, r, r, \cdots]} \in {}^*\mathbb{R} \qquad \mathbb{R} \subset {}^*\mathbb{R}$$

$$\forall r = {}^*r < [n] < [n^2]$$

infinite

$$\frac{1}{[n]} = \left[1, \frac{1}{2}, \frac{1}{3}, \cdots\right]$$

$$\forall r \ (>0) \in \mathbb{R},$$

$$r > \frac{1}{[n]}$$

an infinitesimal.

$$L = \left\{ {}^{*}\varepsilon \cdot z \in {}^{*}\varepsilon \cdot ({}^{*}\mathbb{Z}) \;\middle|\; -\frac{{}^{*}H}{2} \leq {}^{*}\varepsilon \cdot z < \frac{{}^{*}H}{2} \right\},$$

$$L' := \left\{ \varepsilon' \cdot z \in \varepsilon' ({}^{*}\mathbb{Z}) \;\middle|\; -{}^{*}H \frac{H'}{2} \leq \varepsilon' \cdot z < {}^{*}H \frac{H'}{2} \right\}.$$

$$X := \{\, a : \underline{\underline{L}} \to \underline{\underline{L'}},$$

$$\text{internal in } {}^{**}(\quad)\,\}$$

"with double meanings",

$$\Downarrow$$

$$A := \{\, f : X \to {}^{**}\mathbb{C},$$

$$\text{internal in } {}^{**}(\quad)\,\}.$$

$$\delta(a) = \begin{cases} H^{\frac{H^2}{2}} H^{H^2}, & (a = 0) \\ 0, & (a \neq 0). \end{cases}$$

$$\int \mathcal{D}a \exp\left(-2\pi i \int_{-\infty}^{\infty} dk\, a(k)\, b(k)\right) f(a).$$

$$(f * g)(a) := \sum_{b \in X} \varepsilon_0 f(a-b) g(b),$$

(ii) $$F(f * g) = Ff\, Fg,$$

$$F(fg) = F(f) * F(g).$$

$$(D_{+,b} f)(a) := \frac{f(a+\varepsilon' b) - f(a)}{\overset{(-)}{\varepsilon'}},$$

$$\overset{(-)}{\lambda}_b(a) := \frac{\exp(2\pi i \varepsilon' \langle a.b \rangle) - 1}{\varepsilon'},$$

Theorem.

$$F(D_{+,b} f)(a) = \lambda_b(a)(Ff)(a),$$

$$F(\bar{\lambda}_b f)(a) = (D_{+,b}(Ff))(a).$$

$$(f, -D_{+,b}\, g) = -(D_{+,b}\, f, g),$$

$$\lambda_i(x) := \begin{cases} 1, & x = i, \\ 0, & x \neq i. \end{cases}$$

$$[D_{+,\lambda_i}, \lambda_j]\, f(a) = \delta_{ij}\, f(a + \varepsilon' \lambda_i).$$

uncertainty relation for field theory

$$\boxed{\varepsilon_0 := \varepsilon^{\frac{H^2}{2}} \varepsilon'^{H^2}},$$

an infinitesimal of higher degree.

Def. $f \in A$,

$$\boxed{(Ff)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L}{}^{*} \varepsilon\, a(k) b(k)\right) \cdot f(a)},$$

$$\boxed{(\bar{F}f)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(2\pi i \sum_{k \in L}{}^{*} \varepsilon\, a(k) b(k)\right) \cdot f(a)}.$$

Theorem.

(i) $F1 = \delta, \quad F\delta = 1,$

$F\bar{F} = \bar{F}F = id,$

$F^4 = id,$

$$\langle f, g \rangle := \sum_{a \in X} \varepsilon_0 \overline{f(a)}\, g(a),$$

F is unitary w.r.t. $\langle\, ,\rangle$.

Example.

$$f_\xi(a) = \exp\Big(-2\pi\xi \sum_{k \in L} \varepsilon\, a(k)^2\Big),$$

$\xi \in \mathbb{C}$, $\operatorname{Re}\xi > 0$,

$\Rightarrow$

$$(Ff_\xi)(b) = c_\xi(b)\, f_{\frac{1}{\xi}}(b),$$

$$\frac{c_\xi}{(\sqrt{\xi})^{H^2}} \sim 1, \text{ if } b \text{ is finite.}$$

the standard parts are same.

If $\xi = i$, $(e^{\frac{\pi}{4}i})^{H^2} = (-1)^{(\frac{H}{2})^2}$.

X is a group.

$Y < X$,

$$\boxed{Y^{\perp} := \{ b \in X \mid \exp(2\pi i \langle a, b \rangle) = 1, \ \forall a \in Y \}.}$$

<u>**Theorem.**</u> Poisson's summation formula.

$$|Y|^{-\frac{1}{2}} \sum_{a \in Y} f(a) = |Y^{\perp}|^{-\frac{1}{2}} \sum_{b \in Y^{\perp}} Ff(b).$$

Example.

$$|Y|^{-\frac{1}{2}} \sum_{a \in Y} f_\xi(a)$$

$$= |Y^\perp|^{-\frac{1}{2}} \sum_{b \in Y^\perp} C_\xi(b) f_{\frac{1}{\xi}}(b).$$