Critical Points

Vítor Neves
(João Paulo Teixeira, Manfred Wolff,
Maria João Borges, Natália Martins)

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E *is* a Banach space & $f \in C^1(E, \mathbb{R})$

Palais-Smale condition

$(\mathbf{PS}) \quad f'(u_n) \to 0 \ \& \ (f(u_n)) \text{ bounded } \Rightarrow \exists u, (u_{k_n})\, u_{k_n} \to u$

Theorem 1 *If E is separable,* (PS) is equivalent to

$(\mathbf{PS}) f(u)\, is\, finite \ \& \ f'(u) \approx 0 \quad \Rightarrow \quad u\, is\, near{-}standard.$

$$\begin{aligned} K_c &:= f^{-1}(c) \cap f'^{-1}(0) \\ A_\alpha &:= f^{-1}(]-\infty, \alpha]) \\ N_\delta &:= K_c + B_\delta(0). \end{aligned}$$

Lemma 1 *If f verifies (PS), then*

1. $\forall c \in \mathbb{R} \quad K_c$ *is compact.*
2. $\exists b, \hat{\varepsilon} > 0 \quad \forall x \in A_{c+\hat{\varepsilon}} \setminus (A_{c-\hat{\varepsilon}} \cup N_{\frac{\delta}{8}}) \quad \|f'(x)\| \geq b$

The Beauty

Theorem 2 ***(Mountain Pass)***
If

$$f \ \ verifies \ \ (PS)$$

$$\|e\| > \rho > 0 \qquad \& \qquad \max\{f(0), f(e)\} < \inf_{\|x\|=\rho} f(x)$$

$$\Gamma \qquad = \qquad \{\gamma \in C^1([0,1], E) |\ \gamma(0) = 0 \ \& \ \gamma(1) = e\},$$

then

$$\inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)) \quad \textit{is a critical value of } f.$$

The Beast

Lemma 2 ***(Deformation)***

Suppose f satisfies (PS). For all $c \in \mathbb{R}$, $\bar{\varepsilon} > 0$ and any neighborhood O of K_c, there exist $\varepsilon \in \;]0, \bar{\varepsilon}[$ and $\eta \in C([0,1] \times E, E)$ such that

1. $\eta(0, \cdot) = id_E$
2. $\forall x \; |f(x) - c| > \bar{\varepsilon} \;\; \Rightarrow \forall t \in [0,1] \quad \eta(t,x) = x$
3. $\forall t \in [0,1] \quad \eta(t, \cdot)$ *is a homeomorphism*
4. $\forall t \in [0,1], x \in E \quad \|\eta(t,x) - x\| \leq 1$
5. $\forall t \in [0,1], x \in E \quad f(\eta(t,x)) \leq f(x)$
6. $\eta(1, A_{c+\varepsilon} \setminus O) \subseteq A_{c-\varepsilon}$
7. $K_c = \emptyset \quad \Rightarrow \quad \eta(1, A_{c+\varepsilon}) \subseteq A_{c-\varepsilon}$
8. *If f is even, than $\eta(t, \cdot)$ is odd.*

$$0 \; < \; \varepsilon \; < \; \hat{\varepsilon} \; < \min\left\{\bar{\varepsilon}, \frac{b\delta}{32}, \frac{b^2}{2}, \frac{1}{8}\right\}$$

The less beastly beast

Theorem 3 *Suppose f satisfies (PS). If c is not a critical value of f, then*

$$\forall \bar{\varepsilon} > 0 \; \exists \varepsilon \in]0, \bar{\varepsilon}[$$

1. $\forall x \; [\|f(x) - c\| > \bar{\varepsilon} \quad \eta(1,x) = x$
2. $\eta(1, A_{c+\varepsilon}) \subseteq A_{c-\varepsilon}$.

Lemma 3 *Suppose E is an Hilbert space and let U be an open subset of E. Let $f \in C^1(U,\mathbb{R})$ and $x \in ns({}^\star U)$. If $f'(x) \not\approx 0$, then for every $0 < \varepsilon \approx 0$, the following inequality holds:*

$$f(x - \varepsilon f'(x)) < f(x) - \varepsilon \frac{\| f'(x) \|^2}{4}.$$

Lemma 4 *Suppose E is an Hilbert space. Let $\gamma \in C([0,1],E)$ and $f \in C^1(E,\mathbb{R})$ such that*

$$f \circ \gamma \quad \text{is not constant} \qquad \& \qquad 0 < r < \max_{t\in[0,1]} \|f'(\gamma(t))\|.$$

For each function $\eta : [0,1] \to \mathbb{R}_0^+$, define

$$\gamma_\eta(t) := \gamma(t) - \eta(t) f'(\gamma(t))$$

and

$$V_r := \{t \in]0,1[\ | \ \|f'(\gamma(t))\| > r\}.$$

There exists a function $\delta_r \in C([0,1],[0,1])$ such that

$$\begin{array}{ll} \delta_r(0) = \delta_r(1) = 0 & \\ \forall t \in V_r & \delta_r(t) > 0 \\ \forall t \in V_r & f(\gamma_{\delta_r}(t)) < f(\gamma(t)) \end{array}$$

and for all functions $\eta : [0,1] \to \mathbb{R}_0^+$,

$$\forall t \in [0,1] \ [\eta(t) \leq \delta_r(t) \quad \Rightarrow \quad f(\gamma_\eta(t)) \leq f(\gamma(t))].$$

(Very) Tame functionals

Lemma 5

1. *The following conditions are equivalent*
   (a) $\lim_{\|x\|\to+\infty} f(x) = +\infty$ *i.e.* f *is* ***coercive***
   (b) $\forall x \in E \quad x\ is\ infinite \qquad \Rightarrow \qquad f(x) \approx +\infty$
   (c) $\forall x \in E \quad f(x)\ is\ finite \vee f(x) < 0 \quad \Rightarrow \quad x\ is\ finite$
2. *If* E *is finite dimensional*
   (a) *the above conditions are equivalent to*
   $$\forall x \in E \quad [f(x)\ is\ finite \quad \Leftrightarrow \quad x\ is\ finite]$$
   (b) *All coercive functionals verify (PS).*

The Pet

Theorem 4 *If*

$$\begin{array}{rcl} E & is & finite\ dimensional\ (say\ E = \mathbb{R}^n;\ n \in \mathbb{N}) \\ f & is & coercive \\ \|e\| > \rho > 0 & \& & \max\{f(0), f(e)\} < \inf_{\|x\|=\rho} f(x) \\ \Gamma & = & \{\gamma \in C^1([0,1], E) |\ \gamma(0) = 0\ \&\ \gamma(1) = e\}, \end{array}$$

then

$$k_1 \ := \ \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)) \quad is\ a\ critical\ value\ of\ f.$$

$$f \text{ is coercive} \quad \Rightarrow \quad \text{may assume WLOG,}$$

$$For\ some\ fixed\ real\ r > \|e\| \quad \forall \gamma \in \Gamma\ \gamma([0,1]) \ \subseteq\ [-r,r]^n;$$

$$\text{Pick } N \in \mathbb{N} \text{ such that } N > r+1 \text{ and } M \in {}^\star\mathbb{N}_\infty$$

$$h \ := \ \frac{1}{M} \approx 0$$

$$\begin{aligned} \mathcal{R} \ &:= \ {}^\star]-N,N[^n \cap h^\star\mathbb{Z}^n \\ &= \ \{a_i := -N + ih |\ 0 < i < 2NM\}^n \\ &\subseteq \ {}^\star\mathbb{R}^n \end{aligned}$$

$\mathcal{R}$ is hyperfinite

Define

$$e \ \in \ [x_e, x_e + \overrightarrow{h}[:= \prod_{i=1}^{n} [x_{ei}, x_{ei} + h[$$

$$\begin{aligned} \mathcal{P} \ := \ \Big\{ & p({}^\star\mathbb{N}) \mid \ p \in^\star \left((\mathbb{R}^n)^{\mathbb{N}} \right) \ \wedge \ p({}^\star\mathbb{N}) \subseteq \mathcal{R} \ \wedge \ p(1) = 0 \\ & \wedge \left[\exists \omega \in^\star\mathbb{N}\ p(\omega) = x_e \ \wedge \ \forall n \in^\star\mathbb{N}\ [n \geq \omega \Rightarrow p(n) = x_e] \right] \\ & \wedge \left[\ \forall i \in^\star\mathbb{N} \parallel p(i) - p(i+1) \parallel \leq \ \sqrt{n}h \right] \Big\}. \end{aligned}$$

Claim 1: *$\mathcal{P}$ is a hyperfinite set and*

$$c \ := \ \min_{p \in \mathcal{P}} \ \max_{x \in p({}^\star\mathbb{N})} f(x).$$

is well defined.

($\mathcal{P}$ is an internal set of internal parts of the hyperfinite set $\mathcal{R}$)

Claim 2: $c \approx k_1$.

`Primo`

$$k_1 \;<\; c \quad \vee \quad c \;\approx\; k_1$$

Pick $p \in \mathcal{P}$ $(p(\omega) = x_e)$ such that $c = \max_{x \in p({}^\star\mathbb{N})} f(x)$

fill in linearly between each $p(i)$ and $p(i+1)$, i.e., define

$$\gamma(t) \;:=\; \begin{cases} p(i) + \omega\left(t - \frac{i}{\omega}\right)\left(p(i+1) - p(i)\right) & \frac{i}{\omega} \leq t \leq \frac{i+1}{\omega} \\ \\ \qquad (0 \leq i \leq \omega - 1); \end{cases}$$

so that

$$\gamma \in^\star \Gamma \quad \& \quad k_1 \;\leq\; \max_{t \in^\star [0,1]} f(\gamma(t)) \;\approx c$$

`Secundo`

Pick $\varepsilon \in]0,1]$, $\gamma \in \Gamma$ such that

$$k_1 \;=\; \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)) \;\leq\; \max_{t \in [0,1]} f(\gamma(t)) \leq k_1 + \varepsilon$$

recall that

$$\gamma([0,1]) \;\subseteq\; \mathcal{R}.$$

Pick $\Omega \in^* \mathbb{N}_\infty$ such that

$$\left\| \gamma\left(\frac{i+1}{\Omega}\right) - \gamma\left(\frac{i}{\Omega}\right) \right\| \;<\; h \qquad (0 \leq i \leq \Omega - 1)$$

Define $p \in \mathcal{P}$ by

$$\begin{cases} \gamma\left(\frac{i-1}{\Omega}\right) \in [p(i), p(i) + \overrightarrow{h}[& p(i) \in \mathcal{R} \;\;\&\;\; i \in \{1, \cdots, \Omega - 1\} \\ x_e & i \geq \Omega \end{cases}$$

Observe that indeed

$$\|p(i+1) - p(i)\| \;\le\; \sqrt{n}h.$$

Then

$$c \;\le\; \max_{x\in p({}^\star\mathbb{N})} f(x) \;\approx\; \max_{t\in{}^\star[0,1]} f(\gamma(t)) \;<\; k_1 + \varepsilon$$

so that

$$c \;<\; k_1 + \varepsilon;$$

Claim 3: $f^{-1}\Big|_{\mathcal{R}}(c)$ *contains at least one almost critical point*

If this **is not** the case,

$$\forall x \in \mathcal{R} \quad \big[f(x) = c \;\Rightarrow\; f'(x) \not\approx 0\big]$$

Let $p_{min} \in \mathcal{P}$ be such that

$$\max_{x\in p_{min}({}^\star\mathbb{N})} f(x) = \min_{p\in\mathcal{P}} \max_{x\in p({}^\star\mathbb{N})} f(x) = c.$$

Note that

$$[x \in p_{min}({}^\star\mathbb{N}) \;\wedge\; f(x) = c] \quad\Rightarrow\quad [x \not\approx 0 \;\wedge\; x \not\approx e]$$

and let

$$\nu \;:\; \max\{i \in^\star \mathbb{N} |\; f(p_{min}(i)) = c\}.$$

`Primo`

Internally partition $f^{-1}(c) \cap p_{min}({}^\star\mathbb{N})$.

Define recursively

$$\begin{aligned} i_1 &:= \min\{i|\ f(p_{min}(i)) = c\} \\ \overline{i_1} &:= \max\{j|\ \forall i\ [i_1 \le i \le j \Rightarrow f(p_{min}(i)) = c]\} \end{aligned}$$

$$\begin{aligned} i_{\ell+1} &:= \min\{i|\ \overline{i_\ell} < i\ \wedge\ f(p_{min}(i)) = c\} \\ \overline{i_{\ell+1}} &:= \max\{j|\ \forall i\ [\overline{i_\ell} \le i \le j \Rightarrow f(p_{min}(i)) = c]\}. \end{aligned}$$

$$1\ < i_1\ \le\ \overline{i_1}\ <\ \cdots\ < i_\kappa\ \le\ \overline{i_\kappa}\ =\ \nu\ <\ \omega.$$

With

$$C_j\ :=\ \{p_{min}(i)|\ i_j \le i \le \overline{i_j}\} \qquad (1 \le j \le \kappa),$$

we have

$$f^{-1}(c) \cap p_{min}({}^\star\mathbb{N})\ = \dot{\bigcup}_{1 \le j \le \kappa} C_j\ :=\ \mathcal{C},$$

Lemma 6 *Let* $f \in C^1(\mathbb{R}^n, \mathbb{R})$ *and define* $b : \mathbb{R}^n \to \{1, 2, \cdots, n\}$

$$b(x)\ :=\ \max\Big\{k = 1, 2, \cdots, n\ \ | \ \ \Big|\frac{\partial f}{\partial x_k}(x)\Big| = \max\Big\{\Big|\frac{\partial f}{\partial x_i}(x)\Big|\ :\ i = 1, 2, \cdots, n\Big\}\Big\}.$$

If $a \in {}^\star\mathbb{R}^n$ *is such that* $f'(a) \not\approx 0$, *then* $\frac{\partial f}{\partial x_{b(a)}}(a) \not\approx 0$. *Furthermore, if*

$$u_{b(a)} = \begin{cases} -e_{b(a)} & \text{if } \frac{\partial f}{\partial x_{b(a)}}(a) > 0 \\ \ \ e_{b(a)} & \text{if } \frac{\partial f}{\partial x_{b(a)}}(a) < 0 \end{cases}$$

then, $0 \not\approx \partial f_{u_{b(a)}}(a) < 0$.

Repartition the C_j the following way

$$\begin{aligned} m_{j,1} &:= \max\left\{m \middle|\ \forall i\ [i_j \le i \le m \Rightarrow b(p(i)) = b(p(i_j))]\right\} \\ m_{j,\ell+1} &:= \max\left\{m \middle|\ \forall i\ [m_{j,\ell} < i \le m \Rightarrow b(p(i)) = b(p(m_{j,\ell}+1))]\right\} \end{aligned}$$

$$\begin{aligned} C_{j,1} &:= \left\{p(i) \middle|\ i_j \le i \le m_{j,1}\right\} \\ C_{j,\ell} &:= \left\{p(i) \middle|\ m_{j,\ell-1} < i \le m_{j,\ell}\right\}. \end{aligned}$$

For some specific sequence $(\chi_j)_{1\le j\le\kappa}$,

$$\begin{aligned} 1 < i_1 \le m_{1,1} < \ \cdots\ < m_{1,\chi_1} = \overline{i_1} < \cdots \\ < i_\kappa \ \le\ m_{\kappa,1} < \cdots < m_{\kappa,\chi_\kappa} = \overline{i_\kappa} = \nu < \omega, \end{aligned}$$

$$C_j = \dot{\bigcup}_{1\le\ell\le\chi_j} C_{j,\ell}$$

and

$$f^{-1}(c) \cap p_{min}({}^\star\mathbb{N}) = \dot{\bigcup}_{1\le j\le\kappa} \dot{\bigcup}_{1\le\ell\le\chi_j} C_{j,\ell}$$

The function b defined in lemma 6 is constant on each $C_{j,\ell}$ by construction.

`Secundo`

Build $p^- \in \mathcal{P}$ for which $\max f(p^-({}^\star\mathbb{N})) < c$

1. Define a convenient *multi-valued internal* function $P : \{1, \cdots, \omega\} \to \mathcal{R}$

   (a) i. P outside $\mathcal{C}$:

   $$P(i) \;:=\; \begin{cases} p_{min}(i) & p_{min}(i) \notin \mathcal{C} \\ p_{min}(i_j - 1) + hu_{b(p_{min}(i_j))} & i = i_j - 1 \\ p_{min}(\overline{i_j} + 1) + hu_{b(p_{min}(\overline{i_j}))} & i = \overline{i_j} + 1 \\ & (1 \leq j \leq \kappa), \end{cases}$$

   P is 1-2 at the $i_j - 1$ and $\overline{i_j} + 1$.

   ii. P inside C: **First step**

   $$P(i) \;:=\; \begin{cases} p_{min}(i) + hu_{b(p_{min}(i))} & p_{min}(i) \in \mathcal{C} \\ & (1 \leq j \leq \kappa), \end{cases}$$

   The distances between consecutive points outside $\mathcal{C}$ are not changed
   (same happens with the points themselves);
   adding one point at each end of the C_j keeps the passage "into" and "out of" $\mathcal{C}$ within the bound $\sqrt{n}h$.
   when $1 \leq j \leq \kappa$,

   $$\begin{aligned} f(p(i) + hu_{b(p(i))}) &< c & (p(i) \in C_j) \\ f(p(i)) &< c & (p(i) \notin \mathcal{C}) \\ f(p(i_j - 1) + hu_{b(p(i))}) &< f(p(i_j - 1)) < c \\ f(p(\overline{i_j} + 1) + hu_{b(p(\overline{i_j}))}) &< f(p(\overline{i_j} + 1)) < c. \end{aligned}$$

iii. P inside C: **Second step**.

Take care of passages from $C_{j,\ell}$ to $C_{j,\ell+1}$.

$$\begin{cases} p_{min}(i) := \alpha & = \ \textit{last element of } C_{j,\ell} \\ p_{min}(i+1) := \beta & = \ \textit{the first element of } C_{j,\ell+1}. \end{cases}$$

$$u_{b(\alpha)} \perp u_{b(\beta)}$$

because the relevant partial derivatives are S-continuous and non-infinitesimal at α and at β; as their distance is at most $\sqrt{n}h$, α and β are vertices of an interval of the grid $\mathcal{R}$, say I. Define possibly one more image for one or both of the $P(\cdot)$ by:

$$\begin{cases} P(i) = \alpha + hu_{b(\beta))} & \text{if } u_{b(\beta)} \text{ points out of I} \\ P(i+1) = \beta + hu_{b(\alpha))} & \text{if } u_{b(\alpha)} \text{ points out of I} \end{cases}$$

Again

$$f(P(j)) \quad < \quad c \qquad\qquad (j = i, i+1)$$

2. Adequate shifts in i produce a function p^- for which $\max f(p^-({}^\star\mathbb{N})) < c$, an impossibility.

Claim 4: *k_1 is a critical value of f.* □

Theorem 5 ? *Suppose E is (?Hilbert?) separable, $f : E \to \mathbb{R}$ is smooth, verifies the Mountain Pass geometry, verifies (PS), $N \in^* \mathbb{N}_\infty$ and*

$$\exists r \not\in \mathcal{O} \qquad f(S_r(0)) \cap \mathcal{O} = \emptyset$$
$$^\sigma E \ \subseteq \ \mathbb{R}^N \ \subseteq \ ^*E.$$

Then the nonstandard hull of $f_{|\mathbb{R}^N}$ has a critical point.

References

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