Time-Development of Explosions and a Path-Space Measure for Diffusion Process with Repulsive Higer Order Drift

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§1. Time-development of explosion

1. Explosion

Stochastic differential equation

$$dX(t) = f\Big(X(t)\Big)\,dt + dB(t),$$

where

$$X(t) : \text{particle momentum at time } t,$$

$$f\Big(X(t)\Big) : \text{drift}, \qquad \frac{dB(t)}{dt} : \text{random force}$$

If

(i) $f(x)$ grows faster than linear,
(ii) $f(x)$ is repulsive, that is, $xf(x) > 0$,

then the process explodes successively :

$$P(\text{explosion time is finite}) = 1.$$

2. Survival rate

SDE implies forward Fokker-Planck equation (FP-equation) for the probability density $\phi(t,x)$ of a particle momentum x at time t,

$$\frac{\partial}{\partial t}\phi(t,x) = D\frac{\partial^2}{\partial x^2}\phi(t,x) - \frac{\partial}{\partial x}\Big\{f(x)\phi(t,x)\Big\}.$$

Survival rate by time t is given by

$$P(t) := \int_{-\infty}^{\infty} \phi(t,x)dx,$$

so that the time-development of the explosions by

$$1 - P(t).$$

3. Time-development of survival rate

We assume

(A1) $f(x)$ grows faster than linear,

(A2) $\lim_{|x|\to\infty} \frac{f(x)^2}{|f'(x)|} = \infty$,

(A3) some technical conditions.

Thm. 1 *If $f(x)$ is attractive, then*

$$P(t) = 1,$$

that is, no explosions take place.

Thm. 2 *If $f(x)$ is repulsive, then $P(t)$ decreases exponentially in time.*

4. Idea of the proofs

(i) Change the variable from $\phi(t,x)$ to

$$\psi(t,x) := \phi(t,x)\,\mathrm{exp}\Big[-\frac{1}{2D}U(x)\Big]$$

where $U'(x) = f(x)$.

(ii) Then, $\psi(t,x)$ satisfies the imaginary-time Schrödinger equation,

$$-\frac{\partial\psi(t,x)}{\partial t} = H\psi(t,x)$$

where

$$H := -D\frac{\partial^2}{\partial x^2} + V(x),$$
$$V(x) := \frac{f(x)^2}{4D} + \frac{f'(x)}{2}.$$

(iii) Since $V(x) \to \infty$, Hamiltonian H is self-adjoint having CONS of eigenfunctions:

$$Hu_n(x) = E_n u_n(x)$$

$$(E_0 < E_1 < \cdots < E_n < \cdots \to \infty).$$

Expand the initial data as a series with respect to the CONS $\{u_n(x) | n = 0, 1, \cdots\}$. Then,

$$\psi(t, x) = e^{-Ht}\psi(0, x) = \sum_{n=0}^{\infty} c_n e^{-E_n t} u_n(x).$$

(iv) If $f(x)$ is attractive, it is easy to show that

$$E_0 = 0,$$

which implies that $P(t) = 1$.

(v) If $f(x)$ is repulsive, by WKB-approximation,

$$u_0(x) \sim \frac{a_0}{\sqrt{p_0(x)}} \exp\left[\mp \int_0^x p_0(x')dx'\right]$$

with

$$p_0(x) = \left\{\frac{1}{D}\Big(V(x) - E_0\Big)\right\}^{1/2},$$

it is shown that

$$E_0 > 0,$$

which implies that $P(t)$ decreases exponentially. ∎

§2. Solution by path integral

Construct a probability measure over a space of paths s.t.

(i) The solution to the FP-equation is given as a path integral with respect to the measure,

(ii) probabilities are properly distributed not only to the non-exploding paths but also to the exploding ones.

1. Feynman-Kac-Nelson formula

$$\psi(t,x) = \int_{-\infty}^{\infty} dy \psi(0,y) \int \exp\Big[-\int_0^t V(X(s))ds\Big] d\mu^{\mathrm{W}}$$

μ^{W} : Wiener measure pinned at x and y

Hence,

$$\phi(t,x) = \int_{-\infty}^{\infty} dy \phi(0,y) \times \int \exp\Big[\frac{1}{2D}\{U(x)-U(y)\} - \int_0^t V(X(s))ds\Big] d\mu^{\mathrm{W}}$$

(1) FKN-formula gives the information about the measure for the non-exploding paths.

(2) It gives no information for the exploding paths, because $U(x) \to \infty$ as time approaches to their exploding times.

2. Standard analysis vs Nonstandard

To get around this difficulty in standard analysis,

(i) introduce a cutoff N into the momentum space,

(ii) define a probability measure μ_N over a path-space $\mathcal{P}_N$,

(iii) take the limit of μ_N and $\mathcal{P}_N$ as $N \to \infty$.

In nonstandard analysis, these procedures *at a stroke:*

"*cutoff at infinity can be introduced from the beginning* "

(i) discretize the time and the momentum,

(ii) assign a $*$-probability for each $*$-path separately,

(iii) apply Loeb measure theory to derive the standard probability measure.

3. Definitions

$$\varepsilon > 0,\ \delta = \sqrt{2D\varepsilon},\ A = (D/\beta)^{1/2}\Big|\log \beta\varepsilon\Big|.$$

Def. 1 (1) *Let* $\omega : \{0, 1, \cdots, \nu - 1\} \to \{-1,\, 1\}$ *be internal, where* $\nu = [t/\varepsilon]$.

(i) *Sequence* $\{X_k \mid 1 \le k \le \nu\}$:

$$X_k = \begin{cases} X_0 + \sum_{j=0}^{k-1} \omega(j)\delta & \big(\,|X_k| < A\,\big) \\ \pm A & (\,|X_k| \ge A\,). \end{cases}$$

(ii) $X(s, \omega)$: $*$*-polygonal line with vertices*

$$(0, y), (\varepsilon, x_1), \cdots, (\nu\varepsilon, x_\nu).$$

(iii) $\mathscr{P}_A(\,\cdot\,, t : y, 0)$: *the set of* $X(s, \omega)$.

(iv) $X(s, \omega)$ *"living path"* :

$$\forall s \in [0, \nu\varepsilon) \quad |X(s, \omega)| < A.$$

$X(s, \omega)$ *"path dead at infinity"* :

if not.

(2) *If* $X(s,\omega)$ *"living path"*,

$$\mu\Big(X(s,\omega)\Big) = \frac{1}{2^\nu}\exp\Big[\frac{1}{2D}\Big\{U(X(\nu\varepsilon,\omega)) - U(X(0,\omega))\Big\} - \int_0^{\nu\varepsilon} {}^*V(X(s,\omega))ds\Big].$$

If $X(s,\omega)$ *"path dead at infinity"*,

$$\mu\Big(X(s,\omega)\Big) = \frac{1}{2^{k_0}}\exp\Big[\frac{1}{2D}\Big\{U(X(k_0\varepsilon,\omega)) - U(X(0,\omega))\Big\} - \int_0^{k_0\varepsilon} {}^*V(X(s,\omega))ds\Big].$$

where $k_0 = \min\{\, k \mid X(k\varepsilon,\omega) = \pm A\,\}$.

Remark 1 :

$$\exp\left[-\int_0^{\nu\varepsilon} {}^*V(X(s,\omega))ds\right] \leq \exp(-ct).$$

Remark 2 : If $f(x)$ is repulsive,

$$\exp\left[\frac{1}{2D}U(X(\nu\varepsilon,\omega))\right] \text{ is infinite.}$$

If $f(x)$ is attractive,

$$\exp\left[\frac{1}{2D}U(X(\nu\varepsilon,\omega))\right] \text{ is less than 1.}$$

Thm. 3 *The total $*$-measure satisfies*

$$\mu\Big(\mathscr{P}_A(\,\cdot\,,t : y,0)\Big) \simeq 1,$$

namely the standard Loeb measure derived from the nonstandard measure μ is a probability measure.

4. Solution to the FP-equation

Def. 2

$$\mathcal{U}_A(t,x) = \sum_y U(0,y)\mathcal{G}_A(x,t:y,0)2\delta$$

with

$$\mathcal{G}_A(x,t:y,0) = \frac{1}{2\delta}\sum_{X(s,\omega)} \mu\Big(X(s,\omega)\Big),$$

where sum is taken over $\mathscr{P}_A(x,t:y,0)$*.*

Thm. 4 $U(t,x) = \mathrm{st}\,\mathcal{U}_A(t,x)$ *is the solution to the forward Fokker-Planck equation.*