A Model of Quantum Field Theory with a Fundamental Length

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1 Introduction

The relativistic equation of quantum mechanics called Dirac equation

$$i\frac{\hbar}{c}\gamma_\mu\frac{\partial}{\partial x_\mu}\psi(x) - M\psi(x) = 0,\ \ x_0 = ct, x_1 = x, x_2 = y, x_3 = z$$

contains the constants: c (velocity of light): the fundamental constant in the relativity theory, $h = 2\pi\hbar$ (Planck constant): the fundamental constant in quantum mechanics. Dimension: c: [LT^{-1}], h: [ML^{2}T^{-1}].

W. Heisenberg thought that the equation must also contain a constant l with dimension [L].

Arbitrary dimensions are expressed by the combination of c, h and l, e.g., [T] = [L]/[LT^{-1}], [M] = [ML^{2}T^{-1}]/([LT^{-1}][L])

In 1958, Heisenberg with Pauli introduced the equation

$$\frac{\hbar}{c}\gamma_\mu \frac{\partial}{\partial x_\mu}\psi(x) \pm l^2 \gamma_\mu \gamma_5 \psi(x) \bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x) = 0, \tag{1}$$

which is later called the equation of universe. The constant l has the dimension [L] and is called the fundamental length.

Dürr, H.-P.; Heisenberg, W.; Mitter, H.; Schlieder, S.; Yamazaki, K.

Zur Theorie der Elementarteilchen, Z. Naturf. 14a (1959) 441-485

Heisenberg, W., Introduction to the Unified Field Theory of Elementary Particles, John Wiley & Sons (1966)

1965 Shin'ichiro Tomonaga was awarded the Nobel prize for physics.

1967 Heisenberg visited to Japan for the second time (first time 1929).

Heisenberg gave a talk in Kyoto University.

But equation (1) is difficult to solve. So, we consider the following soluble equation having the constant l with the dimension [L]:

$$\left\{\begin{array}{l} \Box\phi(x) + \left(\frac{cm}{\hbar}\right)^2 \phi(x) = 0 \\ \left(i\frac{\hbar}{c}\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right)\psi(x) = 2\gamma_\mu l^2 \psi(x)\phi(x)\frac{\partial\phi(x)}{\partial x_\mu} \end{array}\right. \qquad (2).$$

This equation has no solutions in the axiomatic framework of of Wightman, that is, the field $\psi(x)$ is not an operator-valued tempered distribution.

But $\psi(x)$ is an operator-valued tempered <u>ultrahyperfunction</u>.

The equation (2) has a solution in the framework of

E. Brüning and S. Nagamachi: Relativistic quantum field theory with a fundamental length, J. Math. Phys. 45 (2004) 2199-2231.

2 Wightman axioms

W.I (Relativistic invariance of the state space). There is a physical Hilbert space $\mathcal{H}$ in which a unitary representation $U(a, A)$ of the Poinaré spinor group $\mathcal{P}_0$ acts.
W.II (Spectral property).
W.III (Existence and uniqueness of the vacuum). There exists in $\mathcal{H}$ a unique unit vector Ψ_0 (called the vacuum vector),
W.IV (Fields and temperedness). The components $\phi_j^{(\kappa)}$ of the quantum field $\phi^{(\kappa)}$ are operator-valued generalized functions $\phi_j^{(\kappa)}(x)$ over the Schwartz space $\underline{\mathcal{S}(\boldsymbol{R}^4)}$ with common dense domain of definition $\mathcal{D}$ to all the operaotrs $\phi_j^{(\kappa)}(f)$.
W.V (Cyclicity of the vacuum).
W.VI (Poincaré-covariance of the fields).

W.VII (Locality, or microcausality).

Any two field components $\phi_j^{(\kappa)}(x)$ and $\phi_\ell^{(\kappa')}(y)$ either commute or anti-commute under a spacelike separation of x and y:

If f and g have space-like separeted supports

$$\phi_j^{(\kappa)}(f)\phi_\ell^{(\kappa')}(g)\Psi \mp \phi_\ell^{(\kappa')}(g)\phi_j^{(\kappa)}(f)\Psi = 0$$

for all $\Psi \in \mathcal{D}$. We express

$$\underline{\phi_j^{(\kappa)}(x)\phi_\ell^{(\kappa')}(y)\Psi \mp \phi_\ell^{(\kappa')}(y)\phi_j^{(\kappa)}(x)\Psi = 0 \;\; \text{for} \;\; (x-y)^2 < 0}$$

$$[(x-y)^2 = (x^0-y^0)^2 - (x^1-y^1)^2 - (x^2-y^2)^2 - (x^3-y^3)^2]$$

3 Fundamental length

W.VII (Locality) says that the two events which are space-likely separated are independent. Even if we replace W.VII by a weaker axiom

$$\phi_j^{(\kappa)}(x)\phi_\ell^{(\kappa')}(y)\Psi \mp \phi_\ell^{(\kappa')}(y)\phi_j^{(\kappa)}(x)\Psi = 0 \quad \text{for} \quad (x-y)^2 < -\ell^2 < 0,$$

(the two events which are separated by ℓ are independent), we can prove W.VI

$$\phi_j^{(\kappa)}(x)\phi_\ell^{(\kappa')}(y)\Psi \mp \phi_\ell^{(\kappa')}(y)\phi_j^{(\kappa)}(x)\Psi = 0 \quad \text{for} \quad (x-y)^2 < 0$$

by using other axioms. It is not easy to weaken the condition of locality if the field $\phi_j^{(\kappa)}(x)$ has the localization property. We must introduce generalized functions which have no localization property.

Let $T(-\ell,\ell)=\boldsymbol{R}+i(-\ell,\ell)\subset\boldsymbol{C}$.

$\mathcal{T}(T(-\ell,\ell))\ni f$: holomorphic function in $T(-\ell,\ell)$. Then for $|a|<\ell$, we have

$$\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{a^n}{n!}\delta^{(n)}(x)f(x)dx=\sum_{n=0}^{\infty}\frac{(-a)^n}{n!}f^{(n)}(0)$$

$$=f(-a)=\int_{-\infty}^{\infty}\delta(x+a)f(x)dx.$$

(A): $\Delta_N(x)=\sum_{n=0}^{N}\frac{a^n}{n!}\delta^{(n)}(x)$ converges to $\delta(x+a)=\delta_{-a}(x)$ in $\mathcal{T}(T(-\ell,\ell))'$ as $N\to\infty$. $\operatorname{supp}\Delta_N=\{0\}$, $\operatorname{supp}\delta_{-a}=\{-a\}$.

(B): If $|a|>\ell$, $\Delta_N(x)$ does not converge in $\mathcal{T}(T(-\ell,\ell))'$.

(A) and (B) imply: If $|a|<\ell$ then the distinction between $\{0\}$ and $\{-a\}$ is not clear in $\mathcal{T}(\ell)'$, but if $|a|>\ell$ then the distinction between $\{0\}$ and $\{-a\}$ is clear.

4 Ultrahyperfunction

Hasumi, M., Tohoku Math. J. 13 (1961)

Morimoto, M., Proc. Japan Acad. 51 (1975)

$$T(A) = \boldsymbol{R}^n + iA \subset \boldsymbol{C}^n, \ A \subset \boldsymbol{R}^n.$$

$\boldsymbol{R}^n \supset K$: convex compact

$\mathcal{T}_b(T(K)) \ni f$: f is continuous on $T(K)$, holomorphic in the interior of $T(K)$ and satisfy

$$\|f\|^{T(K),j} = \sup\{|z^p f(z)|; z \in T(K), |p| \leq j\} < \infty, \ j = 0, 1, \ldots.$$

There is a natural mapping for $K_1 \subset K_2$

$$\mathcal{T}_b(T(K_2)) \to \mathcal{T}_b(T(K_1)).$$

Let O be a convex open set in $\boldsymbol{R}^n$. We define

$$\mathcal{T}(T(O)) = \varprojlim \mathcal{T}_b(T(K)),\ K \uparrow O.$$

$\mathcal{T}(T(O))$: Fréchet space

Definition 4.1 tempered ultrahyperfunction is a linear form on the space $\mathcal{T}(T(\boldsymbol{R}^n))$.

$\mathcal{T}(T(\boldsymbol{R}^n))'$: space of tempered ultrahyperfunctions

In the book of I.M. Gel'fand and G.E. Shilov, Generalized functions Vol. 2, (1968), there are function spaces $\mathcal{S}^{1,B}$ and $\mathcal{S}^1 = \lim_{B\to\infty} \mathcal{S}^{1,B} = \lim_{K_1\to\{0\}} \mathcal{T}_b(T(K_1))$, but no space $\lim_{0\leftarrow B} \mathcal{S}^{1,B} = \lim_{\boldsymbol{R}^n\leftarrow K_1} \mathcal{T}_b(T(K_1)) = \mathcal{T}(T(\boldsymbol{R}^n))$.

5 Model

Lagrangian density: Natural unit, $c = \hbar = 1$.

$$L(x) = L_{Ff}(x) + L_{Fb}(x) + L_I(x),$$

$$L_{Ff}(x) = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - \tilde{m})\psi(x),$$

$$L_{Fb}(x) = \frac{1}{2}\{(\partial_\mu \phi(x))^2 - m^2 \phi(x)^2\},$$

$$L_I(x) = 2l^2(\bar{\psi}(x)\gamma^\mu \psi(x))\phi(x)\partial_\mu \phi(x).$$

The field equations

$$\begin{cases} (\Box + m^2)\phi(x) = 0 \\ \left(i\gamma_\mu \dfrac{\partial}{\partial x_\mu} - \tilde{m}\right)\psi(x) = 2\gamma_\mu l^2 \psi(x)\phi(x)\dfrac{\partial \phi(x)}{\partial x_\mu} \end{cases}$$

Quantization – Path integral. Two point function, formally

$$\int \bar{\psi}_\alpha(x_1)\psi_\beta(x_2)\exp i\left\{\int_{\boldsymbol{R}^4} L_I(x)dx\right\} d\mathcal{D}(\psi,\bar{\psi})d\mathcal{G}(\phi)$$

$$\times\left\{\int \exp i\left\{\int_{\boldsymbol{R}^4} L_I(x)dx\right\} d\mathcal{D}(\psi,\bar{\psi})d\mathcal{G}(\phi)\right\}^{-1},$$

$$d\mathcal{G}(\phi) = \exp i\left\{\int_{\boldsymbol{R}^4} L_{Fb}(x)dx\right\}\prod_{x\in\boldsymbol{R}^4} d\phi(x)$$

$$d\mathcal{D}(\psi,\bar{\psi}) = \exp i\left\{\int_{\boldsymbol{R}^4} L_{Ff}(x)dx\right\}\prod_{x\in\boldsymbol{R}^4}\prod_{\alpha=1}^{4}\psi_\alpha(x)\bar{\psi}_\alpha(x).$$

Lattice approximation. M, N: positive integers $L = MN$.

$$\Gamma = \{t = j\Delta; j\in\boldsymbol{Z}, -L < j \le L, \Delta = \sqrt{\pi}/M\} = \boldsymbol{\Delta Z}/(2\sqrt{\pi}N).$$

Linear operator $-\triangle+m^2$ on $\boldsymbol{R}^{\Gamma^4}=\boldsymbol{R}^{4\cdot 2L}$ (difference operator on the lattice Γ^4)

$$-\triangle+m^2:\boldsymbol{R}^{\Gamma^4}\ni\Phi(x)\to-\sum_{\mu=0}^{3}\frac{\Phi(x+e_\mu)+\Phi(x-e_\mu)-2\Phi(x)}{\Delta^2}+m^2\Phi(x)\in\boldsymbol{R}^{\Gamma^4}.$$

Gaussian measure on $\boldsymbol{R}^{4\cdot 2L}$:

$$dG(\Phi)=C\exp\left\{\frac{1}{2}\sum_{y\in\Gamma^4}\left[\sum_{\mu=0}^{3}\frac{\Phi(y+e_\mu)+\Phi(y-e_\mu)-2\Phi(y)}{\Delta^2}\right.\right.$$

$$\left.\left.-m^2\Phi(y)\right]\Delta^4\right\}\prod_{y\in\Gamma^4}d\Phi(y),$$

C: normalization constant $\int dG(\Phi)=1$. The exponent: Euclideanized ($x^0\to -iy^0$, $\boldsymbol{x}\to\boldsymbol{y}$) discretization of Lagrangian $i\int L_{Fb}(x)dx$.

The covariance

$$\int \Phi(y_1)\Phi(y_2)dG(\Phi) = 2(-\triangle + m)^{-1}(y_1, y_2) = 2\mathcal{S}_m(y_1 - y_2)$$

$$\mathcal{S}_m(y_1 - y_2) = (2\pi)^{-4} \sum_{p\in\tilde{\Gamma}^4} e^{ip(y_1-y_2)} \left[\sum_{\mu=0}^{3}(2 - 2\cos p_\mu \Delta)/\Delta^2 + m^2\right]^{-1} \eta^4,$$

$$\tilde{\Gamma} = \{s = j\eta; j \in \boldsymbol{Z}, -L < j \leq L, \eta = \sqrt{\pi}/N\} = \eta\boldsymbol{Z}/(2\sqrt{\pi}M).$$

Nonstandard analysis: $\mathcal{S}_m(y_1 - y_2) \to S_m(y_1 - y_2),\ M, N \to \infty.$

Schwinger function of neutral scalar field of mass m:

$$S_m(y_1 - y_2) = (2\pi)^{-4} \int_{\boldsymbol{R}^4} e^{ip(y_1-y_2)} \left[p^2 + m^2\right]^{-1} d^4p.$$

Measure $dD(\Psi^1, \Psi^2)$ on the Grassmann algebra generated by $\{\Psi^1_\alpha(y), \Psi^2_\alpha(y); \alpha = 1, \ldots, 4,\ y \in \Gamma^4\}$:

$$dD(\Psi^1, \Psi^2) = C' \exp\left\{ - \sum_{y \in \Gamma^4} \Psi^{2T}(y) \left[\sum_{\mu=0}^{3} \gamma^E_\mu \nabla_\mu + \tilde{m} \right] \Psi^1(y) \Delta^4 \right\}$$

$$\times \prod_{y \in \Gamma^4} \prod_{\alpha=1}^{4} d\Psi^1_\alpha(y) d\Psi^2_\alpha(y),$$

$$\Psi^1 = (\Psi^1_1, \ldots, \Psi^1_4)^T,\ \Psi^2 = (\Psi^2_1, \ldots, \Psi^2_4)^T,$$

$$\gamma^E_0 = \gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},\ \gamma^E_j = -i\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix},\ j = 1, 2, 3,$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\nabla_\mu \Psi_k = \begin{cases} \nabla^+ \Psi_k(y) = (\Psi_k(y+e_\mu) - \Psi_k(y))/\Delta & \text{if } \; k = 1, 2, \\ \nabla^- \Psi_k(y) = (\Psi_k(y) - \Psi_k(y-e_\mu))/\Delta & \text{if } \; k = 3, 4. \end{cases}$$

Avoid doubling problem.

$$-L_I(y) = \Psi^{2T}(y) e^{-il^2\Phi(y)^2} \sum_{\mu=0}^{3} \gamma_\mu^E$$

$$\times [P_+ \Psi^1(y+e_\mu)\{e^{-il^2\Phi(y+e_\mu)^2} - e^{-il^2\Phi(y)^2}\}/\Delta$$

$$+P_- \Psi^1(y-e_\mu)\{e^{-il^2\Phi(y)^2} - e^{-il^2\Phi(y-e_\mu)^2}\}/\Delta],$$

$$P_\pm = (1 \pm \gamma_0^E)/2.$$

$L_I(y) \to iL_I(x)$: differences $\to$ derivatives, ($y^0 \to ix^0$, $\boldsymbol{y} \to \boldsymbol{x}$).

Two point Schwinger functions of the interacting fields.

$$\int \Psi^1_\alpha(y_1)\Psi^2_\beta(y_2)\exp\left(\sum_{y\in\Gamma^4} L_I(y)\Delta^4\right) dD(\Psi^1,\Psi^2)dG(\Phi)$$

$$\times\left\{\int \exp\left(\sum_{y\in\Gamma^4} L_I(y)\Delta^4\right) dD(\Psi^1,\Psi^2)dG(\Phi)\right\}^{-1}$$

$$= \int e^{il^2\Phi(y_1)^2}\Psi'^1(y_1)e^{-il^2\Phi(y_2)^2}\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)dG(\Phi)$$

$$= \int \Psi'^1(y_1)\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)\int e^{il^2\Phi(y_1)^2}e^{-il^2\Phi(y_2)^2}dG(\Phi).$$

Change of the variables

$$\Psi^1(y) = e^{il^2\Phi(y)^2}\Psi'^1(y),\ \ \Psi^2(y) = e^{-il^2\Phi(y)^2}\Psi'^2(y).$$

$$\int \Psi'^1(y_1)\Psi'^2(y_2)dD(\Psi'^1, \Psi'^2) = \mathcal{R}_{\tilde{m};\alpha,\beta}(y_1 - y_2) \to R_{\tilde{m};\alpha,\beta}(y_1 - y_2)$$

$$R_{\tilde{m};\alpha,\beta}(y) = \left\{ -\sum_{\mu=0}^{3} \gamma_\mu^E \left(\frac{\partial}{\partial y_\mu} \right) + \tilde{m} \right\}_{\alpha,\beta} S_{\tilde{m}}(y).$$

$$\int e^{il^2\Phi(y_1)^2} e^{-il^2\Phi(y_2)^2} dG(\Phi)$$

$$= \left[(1 - il^2\mathcal{S}_m(0))(1 + il^2\mathcal{S}_m(0)) - l^4\mathcal{S}_m(y_1 - y_2)^2 \right]^{-1/2}.$$

$\mathcal{S}_m(0) \to \infty$ as $N, M \to \infty$. Wick product:

$$: e^{it\Phi(y)} := \sum_{n=0}^{\infty} [: (it\Phi(y))^n : /n!] = e^{-it^2\mathcal{S}_m(0)} e^{it\Phi(y)}.$$

Then we have

$$\int :e^{il^2\Phi(y_1)^2}::e^{-il^2\Phi(y_2)^2}:dG(\Phi)=\left[1-4l^4\mathcal{S}_m(y_1-y_2)^2\right]^{-1/2}.$$

Two point Schwinger function of ψ:

$$\left[1-4l^4\mathcal{S}_m(y_1-y_2)^2\right]^{-1/2}\mathcal{R}_{\tilde{m};\alpha,\beta}(y_1-y_2),$$

$$\rightarrow\left[1-4l^4S_m(y_1-y_2)^2\right]^{-1/2}R_{\tilde{m};\alpha,\beta}(y_1-y_2)$$

Two point Wightman function

$$\left[1-4l^4D_m^{(-)}(x_0-i\epsilon,\boldsymbol{x})^2\right]^{-1/2}=\left[1-4l^4S_m(ix_0+\epsilon,\boldsymbol{x})^2\right]^{-1/2}.$$

$$D_m^{(-)}(x)=D_m^{(-)}(x_0,\boldsymbol{x}):=\lim_{\epsilon\rightarrow 0}D_m^{(-)}(x_0-i\epsilon,\boldsymbol{x}).$$

$$|D_m^{(-)}(x_0 - i\epsilon, \boldsymbol{x})| \leq (2\pi\epsilon)^{-2}, \quad \epsilon^2 D_m^{(-)}(-i\epsilon, \mathbf{o}) \to (2\pi)^{-2} \ (\epsilon \to 0).$$

If $\epsilon > \sqrt{2}l/(2\pi)$, then $|4l^4 D_m^{(-)}(x_0 - i\epsilon, \boldsymbol{x})^2| < 1$ and

$$\left[1 - 4l^4 D_m^{(-)}(z_0, \boldsymbol{x})^2\right]^{-1/2}, \ \mathrm{Im}\, z_0 > \sqrt{2}l/(2\pi)$$

defines a ultrahyperfunction W by

$$W(f) = \int_{\boldsymbol{R}^4} \left[1 - 4l^4 D_m^{(-)}(x_0 - i\epsilon, \boldsymbol{x})^2\right]^{-1/2} f(x_0 - i\epsilon, \boldsymbol{x}) dx$$

for $f \in \mathcal{T}(T(O_s))$,

$$O_s = \{x \in \boldsymbol{R}^4; \|x\| < s\}$$

for some $s > \sqrt{2}l/(2\pi)$. $\sqrt{2}l/(2\pi)$ is the fundamental length, i.e., two events within the distance $\sqrt{2}l/(2\pi)$ cannot be distinguished.

6 Continuous limit

$$\Gamma = \{x = j\Delta; j \in \boldsymbol{Z}, -L < j \leq L, \Delta = \sqrt{\pi}/M\} = \Delta\boldsymbol{Z}/(2\sqrt{\pi}N),$$

$$\tilde{\Gamma} = \{p = j\eta; j \in \boldsymbol{Z}, -L < j \leq L, \eta = \sqrt{\pi}/N\} = \eta\boldsymbol{Z}/(2\sqrt{\pi}M).$$

$$-\triangle + m^2 : \boldsymbol{R}^{\Gamma^4} \ni \Phi(x) \to -\sum_{\mu=0}^{3} \frac{\Phi(x+e_\mu) + \Phi(x-e_\mu) - 2\Phi(x)}{\Delta^2} + m^2\Phi(x) \in \boldsymbol{R}^{\Gamma^4}.$$

Lattice Fourier transformation:

$$\Phi(x) = (2\pi)^{-2} \sum_{p \in \tilde{\Gamma}^4} e^{ipx}\tilde{\Phi}(p)\eta^4.$$

$$\tilde{\Phi}(p) \to \sum_{\mu=0}^{3} \left(\frac{-e^{ip_\mu \Delta} - e^{-ip_\mu \Delta} + 2}{\Delta^2} + m^2 \right) \tilde{\Phi}(p) = \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right) \tilde{\Phi}(p).$$

$$(-\triangle + m^2)^{-1} : (2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} e^{ip(x-y)} \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta^4.$$

$$A(\boldsymbol{p})^2 = \sum_{\mu=1}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2, \;\; A(\boldsymbol{p}) > 0$$

$$\sum_{p \in \tilde{\Gamma}} \frac{e^{ixp}}{(2 - 2\cos p\Delta)/\Delta^2 + A(\boldsymbol{p})^2} \eta = \sum_{p \in \tilde{\Gamma}} \frac{e^{ixp}}{(2 - e^{ip\Delta} - e^{-ip\Delta})/\Delta^2 + A(\boldsymbol{p})^2} \eta$$

$$= \sum_{p \in \tilde{\Gamma}} \frac{e^{i(x+\Delta)p}}{(2e^{ip\Delta} - e^{i2p\Delta} - 1)/\Delta^2 + e^{ip\Delta} A(\boldsymbol{p})^2} \eta,$$

$$z = e^{ip\Delta}, \;\; z^2 - (2 + \Delta^2 A(\boldsymbol{p})^2) z + 1 = 0,$$

$$z = z_\pm = \frac{2 + \Delta^2 A(\boldsymbol{p})^2 \pm \Delta A(\boldsymbol{p}) \sqrt{4 + \Delta^2 A(\boldsymbol{p})^2}}{2}.$$

$z_+ > 1 > z_- > 0$, $z_+ z_- = 1$.

$$\frac{1}{(2e^{ip\Delta} - e^{i2p\Delta} - 1)/\Delta^2 + e^{ip\Delta} A(\boldsymbol{p})^2} = \frac{-\Delta^2}{z^2 - (2 + \Delta^2 A(\boldsymbol{p})^2) z + 1}$$

$$= \frac{-\Delta^2}{(z - z_+)(z - z_-)} = \frac{\Delta}{z_+ - z_-} \left[\frac{\Delta}{z - z_-} - \frac{\Delta}{z - z_+} \right].$$

$$\sum_{p\in\tilde\Gamma}\frac{\Delta e^{i(x+\Delta)p}}{e^{ip\Delta}-z_-}\eta=\sum_{p\in\tilde\Gamma}\frac{\Delta e^{ixp}}{1-e^{-ip\Delta}z_-}\eta=\sum_{k=0}^{\infty}\sum_{p\in\tilde\Gamma}\Delta e^{i(x-k\Delta)p}z_-^k\eta$$

$= 2\pi z_-^{x/\Delta} = 2\pi z_+^{-x/\Delta}$, for $x \geq 0$. $\quad \sum_{p\in\tilde\Gamma}\frac{\Delta e^{i(x+\Delta)p}}{e^{ip\Delta}-z_+}\eta = 2\pi z_+^{x/\Delta}$ for $x < 0$.

$$\int_{-\sqrt{\pi}M}^{\sqrt{\pi}M}\frac{\Delta e^{ixp}e^{i\Delta p}}{e^{ip\Delta}-z_-}dp=\int_{|z|=1}\frac{z^{x/\Delta}}{z-z_-}\frac{dz}{i}=2\pi z_-^{x/\Delta}.$$

$$\sum_{p\in\tilde\Gamma}\frac{e^{ixp}}{(2-2\cos p\Delta)/\Delta^2+A(\boldsymbol{p})^2}\eta=\frac{2\pi\Delta z_+^{-|x|/\Delta}}{z_+-z_-}$$

$$=\frac{2\pi(1+\Delta A(\boldsymbol{p})[\sqrt{4+\Delta^2A(\boldsymbol{p})^2}/2+\Delta A(\boldsymbol{p})/2])^{-|x|/\Delta}}{A(\boldsymbol{p})\sqrt{4+\Delta^2A(\boldsymbol{p})^2}}.$$

Note: for $-\sqrt{\pi}M \le p_\mu \le \sqrt{\pi}M$,

$$|\boldsymbol{p}|^2 = \sum_{\mu=1}^{3} |p_\mu|^2 \ge \sum_{\mu=1}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} \ge 4/\pi^2 \sum_{\mu=1}^{3} |p_\mu|^2 = 4/\pi^2 |\boldsymbol{p}|^2.$$

Let $M, N \in {}^*\boldsymbol{N} \backslash \boldsymbol{N}$, $M_0 = \sqrt{M}$. If $|\boldsymbol{p}| \le M_0$,

$$A(\boldsymbol{p}) \le \sqrt{|\boldsymbol{p}|^2 + m^2} \le \sqrt{M_0^2 + m^2},\ \Delta A(\boldsymbol{p}) \le \sqrt{\pi}\sqrt{1/M + m^2/M^2} \approx 0$$

$$\delta = \Delta A(\boldsymbol{p})[\sqrt{4 + \Delta^2 A(\boldsymbol{p})^2}/2 + \Delta A(\boldsymbol{p})/2] \approx 0.$$

$$\frac{(1 + \Delta A(\boldsymbol{p})[\sqrt{4 + \Delta^2 A(\boldsymbol{p})^2}/2 + \Delta A(\boldsymbol{p})/2])^{-|x|/\Delta}}{A(\boldsymbol{p})\sqrt{4 + \Delta^2 A(\boldsymbol{p})^2}} = \frac{[(1+\delta)^{1/\delta}]^{-\delta|x|/\Delta}}{A(\boldsymbol{p})\sqrt{4 + \Delta^2 A(\boldsymbol{p})^2}}$$

$$= \frac{e_*^{-A(\boldsymbol{p})[\sqrt{4+\Delta^2 A(\boldsymbol{p})^2}/2|x| + \Delta A(\boldsymbol{p})/2]|x|}}{A(\boldsymbol{p})\sqrt{4 + \Delta^2 A(\boldsymbol{p})^2}} \begin{cases} \approx & e^{-A(\boldsymbol{p})|x|}/2A(\boldsymbol{p}) \quad \text{for some} \\ \le & 2^{-A(\boldsymbol{p})|x|}/2A(\boldsymbol{p}) \quad\quad e_* \approx e \end{cases}$$

$$(2\pi)^{-4}\sum_{p_0\in\tilde{\Gamma}} e^{ipx}\left(\sum_{\mu=0}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2}+m^2\right)^{-1}\eta=(2\pi)^{-3}\frac{e^{i\boldsymbol{px}}e_{**}^{-A(\boldsymbol{p})|x_0|}}{2A(\boldsymbol{p})},$$

and $e_{**}\approx e$.

Let $M_1>0$ be finite. If $|p_\mu|\le M_1$ then $\dfrac{2-2\cos p_\mu\Delta}{\Delta^2}\approx p_\mu^2$ and

$$\frac{e_*^{-A(\boldsymbol{p})|x_0|}}{2A(\boldsymbol{p})}\approx\frac{e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}.$$

$$\left|\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\le M_1}\frac{e^{i\boldsymbol{px}}e_*^{-A(\boldsymbol{p})|x_0|}}{2A(\boldsymbol{p})}\eta^3-\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\le M_1}\frac{e^{i\boldsymbol{px}}e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}\eta^3\right|\approx 0.$$

Note:

$$\frac{(1+\Delta A(\boldsymbol{p})[\sqrt{4+\Delta^2 A(\boldsymbol{p})^2}/2+\Delta A(\boldsymbol{p})/2])^{-|x|/\Delta}}{A(\boldsymbol{p})\sqrt{4+\Delta^2 A(\boldsymbol{p})^2}}$$

is decreasing function of $A(\boldsymbol{p})$.

If $|\boldsymbol{p}| \geq M_0$ then

$$\left|(2\pi)^{-4}\sum_{p_0\in\tilde{\Gamma}} e^{ipx}\left(\sum_{\mu=0}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2}+m^2\right)^{-1}\eta\right|$$

$$\leq (2\pi)^{-3}\frac{2^{-A(\boldsymbol{p})|x_0|}}{2A(\boldsymbol{p})}\,|_{|\boldsymbol{p}|=M_0} \leq (2\pi)^{-3}2^{-2M_0|x_0|/\pi}\frac{1}{4M_0/\pi}.$$

$$\left|(2\pi)^{-4}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\geq M_0}\sum_{p_0\in\tilde{\Gamma}} e^{ipx}\left(\sum_{\mu=0}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2}+m^2\right)^{-1}\eta^4\right|$$

$$\leq (2\pi)^{-3}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\geq M_0}\frac{2^{-2M_0|x_0|/\pi}}{4M_0/\pi}\eta^3 \leq (2\pi)^{-3/2}M^3\frac{2^{-2M|x_0|/\pi}}{4M/\pi}\approx 0.$$

Since

$$\frac{e_{**}^{-A(\boldsymbol{p})|x_0|}}{2A(\boldsymbol{p})}\leq\frac{2^{-A(\boldsymbol{p})|x_0|}}{2A(\boldsymbol{p})}\leq\frac{2^{-2|\boldsymbol{p}||x_0|/\pi}}{4|\boldsymbol{p}|/\pi},$$

for any standard $\epsilon > 0$, there exists a finite M_1 such that

$$\left|(2\pi)^{-3}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,M_1\leq|\boldsymbol{p}|\leq M_0}\frac{e_{**}^{i\boldsymbol{p}\boldsymbol{x}}e^{-A(\boldsymbol{p})|x_0|}}{2A(\boldsymbol{p})}\eta^3\right| < \epsilon.$$

$$\forall \epsilon > 0 \ \exists M_1 \left| (2\pi)^{-4} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, |\boldsymbol{p}| \geq M_1} \sum_{p_0 \in \tilde{\Gamma}} e^{ipx} \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta^4 \right| < \epsilon.$$

Also

$$\forall \epsilon > 0 \ \exists M_1 \ \left| (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, |\boldsymbol{p}| \geq M_1} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2 + m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2 + m^2}} \eta^3 \right| < \epsilon.$$

Proposition

$$(2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} e^{ipx} \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta^4$$

$$\approx (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2 + m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2 + m^2}} \eta^3.$$

Proposition

$$(2\pi)^{-3}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3}\frac{e^{i\boldsymbol{p}\boldsymbol{x}}e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}\eta^3\approx(2\pi)^{-3}\int_{\boldsymbol{p}\in\boldsymbol{R}^3}\frac{e^{i\boldsymbol{p}\boldsymbol{x}}e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}d\boldsymbol{p}.$$

Proposition

$$(2\pi)^{-4}\sum_{p\in\tilde{\Gamma}^4}e^{ipx}\left(\sum_{\mu=0}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2}+m^2\right)^{-1}\eta^4$$

$$\approx(2\pi)^{-3}\int_{\boldsymbol{R}^3}\frac{e^{i\boldsymbol{p}\boldsymbol{x}}e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}d\boldsymbol{p}=(2\pi)^{-4}\int_{\boldsymbol{R}^4}\frac{e^{ipx}}{p^2+m^2}dp$$